Večkratni integral

Dvojni integral

Fubinijev izrek:
Če je f integrabilna na $[a,b]\times[c,d]$ in $y \to f(x,y)$ integrabilna na $[c,d]$ za vsak $x \in [a,b]$:

$$\iint\limits_{[a,b]\times[c,d]} f(x,y)\ dxdy = \int\limits_a^b \left(\int\limits_c^d f(x,y)\ dy \right) dx$$

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$$\iint\limits_{[a,b]\times[c,d]} f(x,y)\ dxdy = \int\limits_c^d \left(\int\limits_a^b f(x,y)\ dx \right) dy$$

Naloge

1. Izračunaj dvojni integral $\iint\limits_Q xy+y^2\ dxdy$, kjer je Q štirikotnik z oglišči $A(0,0)$, $B(4,0)$, $C(4,2)$ in $D(0,2)$.

2. Izračunaj integral $\iint\limits_D xy\ dxdy$, kjer je D omejeno območje med krivujama $y = 2x^2$ in $y = 2$.

3. Izračunaj integral $\iint\limits_D \frac{x}{x^2+y^2}\ dxdy$, $D = \{(x,y) \in \mathbb{R}^2 \mid x\tan x \leq y \leq x,\ 0 < x < \frac{\Pi}{4}\}$.

4. Izračunaj integral $\iint\limits_{\substack{y<2x \\ x<2y}} \frac{1}{xy}\ dxdy$

5. Na dva načina (dva vrstna reda integriranja) izračunaj integral $\iint\limits_{x>2y>0} e^{-x}\ dxdy$.

6. Izračunaj integral $\int\limits_0^\infty \int\limits_0^\infty e^{-(x+y)^2}\ dxdy$

7. Zamenjaj vrstni red integriranja:

a) $\int\limits_0^\infty dx \int\limits_0^{x+1} f(x,y)\ dy$

b) $\int\limits_{-2}^2 dx \int\limits_0^{4-x^2} f(x,y)\ dy$

c) $\int_{-1}^{1} dx \int_{1-x^2}^{2-x^2} f(x,y)\ dy$

d) $\int_{2}^{4} dx \int_{x}^{2x-3} f(x,y)\ dy$

Uvedba novih spremenljivk v dvojni integral

$$\iint_D f(x,y)\ dxdy = \iint_\Delta f(g(u,v),h(u,v)) \cdot |J|\ dudv;$$

$J = \begin{vmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{vmatrix}$, Δ je območje D, zapisano v novih spremenljivkah

Polarne koordinate

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$

$r > 0$, $0 \le \varphi \le 2\pi$, $J = r$

Premaknjene polarne koordinate

$$x = a + r\cos\varphi$$
$$y = b + r\sin\varphi$$

$r > 0$, $0 \le \varphi \le 2\pi$, $J = r$

Naloge

8. Izračunaj integral $\iint_D x\ dxdy$, kjer je D območje v prvem kvadrantu ($x > 0$, $y > 0$) in je omejen s krivuljami $x = y$, $x = 9y$, $xy = 1$ in $xy = 4$.

9. Izračunaj integral $\iint_{\substack{1<x^5y^2<2 \\ x^3y>3}} \frac{1}{x^4y^2}\ dxdy$.

10. Izračunaj integral $\int_0^\infty \int_0^\infty (x-y)^2 e^{-(x+y)^2}\ dydx$

11. Izračunaj $\iint_{\substack{x,y>0 \\ x^2+y^4<4}} \frac{xy}{(1+(x^2+y^2)^2)^2}\ dxdy$

12. Izračunaj integral $\iint_D \sqrt{(x+1)^2+(y+1)^2}\ dxdy$, kjer je D krog s središčem v izhodišču in polmerom $\sqrt{2}$.

Trojni integral

$D \ldots$ območje v prostoru $D: \ h(x,y) \leq z \leq g(x,y)$

$$\iiint\limits_D f(x,y,z)\ dxdydz = \iint\limits_\Delta \left(\int_{h(x,y)}^{g(x,y)} f(x,y,z)\ dz \right) dxdy$$

Volumen telesa D: $V(D) = \iiint\limits_D 1\ dxdydz$

Naloge

13. Izračunaj volumen telesa, ki ga omejujejo ploskve $z = 0$, $z = y$, $y = 1 - x^2$ in $y = 2 - 2x^2$.

14. Izračunaj integral $\iiint\limits_{\substack{x<2 \\ x^2<y<x^3 \\ y<z<xy}} \frac{1}{z}\ dxdydz$.

15. Izračunaj $\iiint\limits_V (x+y+z+1)^{-3}\ dxdydz$, kjer je

$$V := \{(x,y,z) \in \mathbb{R}^3 \mid x,y,z \geq 0,\ x+y+z \leq 1\}.$$

Uvedba novih spremenljivk v trojni integral

$$\iiint\limits_D f(x,y,z)\ dxdydz = \iiint\limits_\Delta f(g(u,v,w),h(u,v,w),k(u,v,w)) \cdot |J|\ dudvdw;$$

$$J = \begin{vmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & \frac{\partial g}{\partial w} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial w} \\ \frac{\partial k}{\partial u} & \frac{\partial k}{\partial v} & \frac{\partial k}{\partial w} \end{vmatrix}, \qquad \Delta \text{ je območje } D, \text{ zapisano v novih spremenljivkah}$$

Cilindrične koordinate

$$\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \\ z &= z \end{aligned}$$

$z \in \mathbb{R}$, $r > 0$, $0 \leq \varphi \leq 2\pi$, $J = r$

Premaknjene cilindrične koordinate

$$\begin{aligned} x &= a + r\cos\varphi \\ y &= b + r\sin\varphi \\ z &= z \end{aligned}$$

$z \in \mathbb{R}$, $r > 0$, $0 \leq \varphi \leq 2\pi$, $J = r$

Sferične koordinate

$$\begin{aligned} x &= r\cos\varphi\cos\vartheta \\ y &= r\sin\varphi\sin\vartheta \\ z &= r\cos\varphi \end{aligned}$$

$r > 0,\ 0 \le \varphi \le 2\pi,\ -\frac{\pi}{2} \le \vartheta \le \frac{\pi}{2},\ J = r^2\cos\vartheta$

Premaknjene sferične koordinate

$$\begin{aligned} x &= a + r\cos\varphi\cos\vartheta \\ y &= b + r\sin\varphi\sin\vartheta \\ z &= r\cos\varphi \end{aligned}$$

$r > 0,\ 0 \le \varphi \le 2\pi,\ -\frac{\pi}{2} \le \vartheta \le \frac{\pi}{2},\ J = r^2\cos\vartheta$

Naloge

16. Izračunaj prostornino telesa

$$T = \left\{ (x,y,z) \in \mathbb{R}^3 \mid 1 \le x^2 + y^2 \le 4,\ 0 \le z \le \frac{4}{x^2+y^2} \right\}.$$

17. Izračunaj prostornino telesa

$$T = \left\{ (x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 \le 2x,\ 0 \le z \le xy,\ y \ge 0 \right\}.$$

18. Izračunaj prostornino telesa

$$T = \left\{ (x,y,z) \in \mathbb{R}^3 \mid 1 \le x^2 + y^2 + z^2 \le 16,\ x^2 + y^2 \le z^2,\ z \ge 0 \right\}.$$

19. Izračunaj prostornino telesa

$$T = \left\{ (x,y,z) \in \mathbb{R}^3 \mid 1 \le x^2 + y^2 + z^2 \le 16,\ x^2 + y^2 \le z^2 \right\}.$$

Rešitve

1 $16 + \frac{32}{3}$

2 0

3 $\frac{\pi^2}{32}$

4 Integral ne obstaja.

5 $\frac{1}{2}$

6 $\frac{1}{2}$

7 a) $\int_0^1 dy \int_0^\infty f(x,y)\ dy + \int_1^\infty dy \int_{y-1}^\infty f(x,y)\ dx$

b) $\int_0^4 dy \int_{-\sqrt{4-y}}^{\sqrt{4-y}} f(x,y)\ dx$

c) $\int_0^1 dy \int_{-1}^{-\sqrt{1-y}} f(x,y)\ dx + \int_0^1 dy \int_{\sqrt{1-y}}^1 f(x,y)\ dx + \int_1^2 dy \int_{-\sqrt{2-y}}^{\sqrt{2-y}} f(x,y)\ dx$

d) $\int_1^2 dy \int_{\frac{y+3}{2}}^2 f(x,y)\ dx + \int_2^4 dy \int_{\frac{y+3}{2}}^y f(x,y)\ dx + \int_4^5 dy \int_{\frac{y+3}{2}}^4 f(x,y)\ dx$

8 $\frac{28}{3}$

9 $\frac{1}{3}\ln 2$

10 $\frac{1}{6}$

11 $\frac{2}{17}$

12 $\frac{64\sqrt{2}}{9}$

13 $\frac{8}{5}$

14 $\frac{4}{3}\ln 2 - \frac{23}{144}$

15 $-\frac{5}{16} + \frac{1}{2}\ln 2$

16 $8\pi \ln 2$

17 $\frac{2}{3}$

18 $21(2-\sqrt{2})\pi$

19 $42(2-\sqrt{2})\pi$