DVOJNI IN TROJNI INTEGRAL

Fubinijev izrek:

Če je f integrabilna na $[a,b]\times[c,d]$ in $y \mapsto f(x,y)$ integrabilna na $[c,d]$ za vsak $x \in [a,b]$:

$$\iint\limits_{[a,b]\times[c,d]} f(x,y)\,dx\,dy = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

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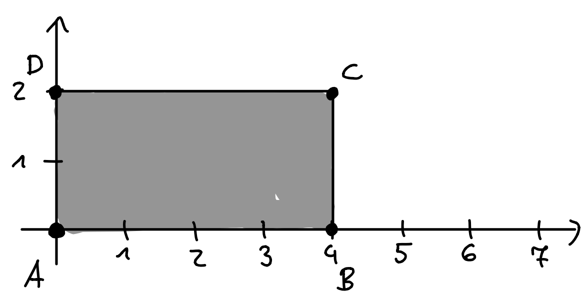
Naloga: Izračunaj dvojni integral $\iint\limits_Q xy+y^2\,dx\,dy$, kjer je Q štirikotnik z oglišči $A(0,0)$, $B(4,0)$, $C(4,2)$ in $D(0,2)$.



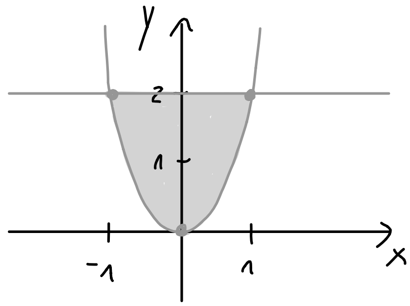
$$\iint\limits_Q (xy+y^2)\,dx\,dy = \int_0^4 \left(\int_0^2 xy + y^2\,dy \right) dx =$$

$$= \int_0^4 \left(x\frac{y^2}{2} + \frac{y^3}{3} \right) \Big|_0^2 \, dx = \int_0^4 \left(2x + \frac{8}{3} \right) dx =$$

$$= \cancel{2}\frac{x^2}{\cancel{2}} + \frac{8}{3}x \Big|_0^4 = \underline{\underline{16 + \frac{32}{3}}}$$

Naloga: $\iint_D xy\,dxdy$, D je omejen s krivuljama $y=2x^2$ in $y=2$.

Narišemo:



$\Rightarrow \quad x \in [-1,1],$
$y \in [2x^2, 2]$

$$\Rightarrow \iint_D xy\,dxdy \underset{\substack{\uparrow \\ y \mapsto xy \text{ integrabilna}}}{=} \int_{-1}^{1}\left(\int_{2x^2}^{2} xy\,dy\right)dx = \int_{-1}^{1} x\frac{y^2}{2}\Big|_{2x^2}^{2}\,dx =$$

$$= \int_{-1}^{1} x\cdot(2-2x^4)\,dx = 2\int_{-1}^{1} x - x^5\,dx =$$

$$= 2\cdot\left(\frac{x^2}{2}-\frac{x^6}{6}\right)\Big|_{-1}^{1} = 2\cdot\left(\frac{1}{2}-\frac{1}{6}-\frac{1}{2}+\frac{1}{6}\right) =$$

$$= 2\cdot 0 = \underline{\underline{0}}$$

Naloga: $\iint_D \frac{x}{x^2+y^2}\,dxdy$, $D=\{(x,y)\in\mathbb{R}^2 \mid x\tan x \le y \le x,\ 0<x<\frac{\pi}{4}\}$

$$\iint_D \frac{x}{x^2+y^2}\,dxdy = \int_0^{\frac{\pi}{4}}\left(\int_{x\tan x}^{x}\frac{x}{x^2+y^2}\,dy\right)dx = \int_0^{\frac{\pi}{4}}\left(\int_{x\tan x}^{x}\frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2}\,dy\right)dx =$$

$t=\frac{y}{x}$
$dt=\frac{1}{x}dy$
$dy = x\,dt$

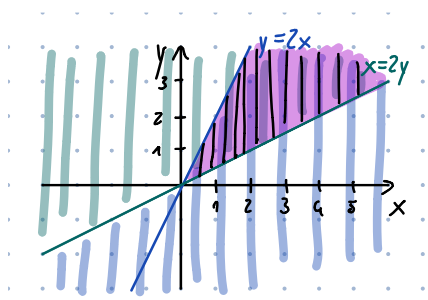
$$= \int_0^{\frac{\pi}{4}}\left(\int_{\tan x}^{1}\frac{\cancel{\frac{1}{x}}}{1+t^2}\cancel{x}\,dt\right)dx =$$

$$= \int_0^{\frac{\pi}{4}} \arctan t\Big|_{\tan x}^{1}\,dx = \int_0^{\frac{\pi}{4}}(\arctan 1 - x)\,dx =$$

$$= \int_0^{\frac{\pi}{4}}\frac{\pi}{4}\,dx - \frac{x^2}{2}\Big|_0^{\frac{\pi}{4}} = \frac{\pi}{4}x\Big|_0^{\frac{\pi}{4}} - \frac{1}{2}\cdot\frac{\pi^2}{4^2} + 0 = \frac{\pi^2}{16}-\frac{\pi^2}{32} = \underline{\underline{\frac{\pi^2}{32}}}$$

N: Izračunajte $\displaystyle\iint\limits_{\substack{y<2x\\ x<2y}} \frac{1}{xy}\,dx\,dy$.

Najprej narišimo območje $\begin{matrix} y<2x \\ x<2y \end{matrix}$



$\Rightarrow x \in [0,\infty) \rightarrow y \in [\frac{x}{2}, 2x]$

$$\iint\limits_{\substack{y<2x\\ x<2y}} \frac{1}{xy}\,dx\,dy = \int_0^\infty \left(\int_{\frac{x}{2}}^{2x} \frac{1}{xy}\,dy \right) dx =$$

$$= \int_0^\infty \frac{1}{x} \left(\int_{\frac{x}{2}}^{2x} \frac{1}{y}\,dy \right) dx =$$

$$= \int_0^\infty \frac{1}{x} \left(\ln|y|\right)\Big|_{\frac{x}{2}}^{2x} dx =$$

$$= \int_0^\infty \frac{1}{x} \left(\ln 2x - \ln \tfrac{1}{2}x \right) dx =$$

$$= \int_0^\infty \frac{1}{x} \ln \frac{2\cancel{x}}{\frac{\cancel{x}}{2}}\,dx = \int_0^\infty \frac{1}{x} \cdot \ln 4\,dx =$$

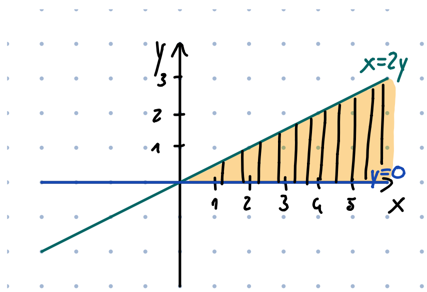
$$= \ln 4 \cdot \int_0^\infty \frac{1}{x}\,dx = \ln 4 \cdot \ln|x| \Big|_{x=0}^{\infty} = \infty$$

$\Rightarrow$ ta integral ne obstaja

To je <u>primer</u> integrala na <u>neomejenem območju</u>, ki <u>ne konvergira</u>!

DN: Zamenjaj vrstni red integriranja!

U: Izračunajte $\iint\limits_{x>2y>0} e^{-x}\,dx\,dy$.



$$\iint\limits_{x>2y>0} e^{-x}\,dx\,dy = \int_0^\infty\left(\int_0^{\frac{x}{2}} e^{-x}\,dy\right)dx =$$

$$= \int_0^\infty e^{-x}\cdot (y)\Big|_0^{\frac{x}{2}}\,dx = \int_0^\infty e^{-x}\cdot\left(\frac{x}{2}-0\right)dx =$$

$$= \frac{1}{2}\int_0^\infty x\cdot e^{-x}\,dx = \frac{1}{2}\left(-xe^{-x}\Big|_0^\infty + \int_0^\infty e^{-x}\,dx\right) =$$

$$= \frac{1}{2}\Gamma(2)$$

$u = x \quad dv = e^{-x}dx$

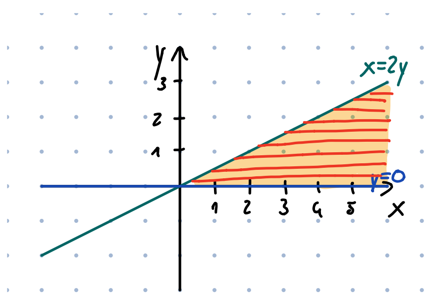
$du = dx \quad v = -e^{-x}$

$$= \frac{1}{2}\left(0 + 0 - e^{-x}\Big|_0^\infty\right) =$$

$$= \frac{1}{2}\left(-0-(-1)\right) = \underline{\underline{\frac{1}{2}}}$$

To je primer integrala na neomejenem območju, ki konvergira!

* Izračunajte z obrnjenim vrstnim redom integriranja.



$y \in [0,\infty) \rightarrow x \in [2y, \infty)$

pri določenem y!

$$\iint\limits_{x>2y>0} e^{-x}\,dx\,dy = \int_0^\infty\left(\int_{2y}^\infty e^{-x}\,dx\right)dy =$$

$$= \int_0^\infty (-e^{-x})\Big|_{2y}^\infty\,dy =$$

$$= -\int_0^\infty (0 - e^{-2y})\,dy = \int_0^\infty e^{-2y}\,dy =$$

$$= \frac{e^{-2y}}{-2}\Big|_0^\infty = \left(0 - \frac{e^0}{-2}\right) = \underline{\underline{\frac{1}{2}}}$$

N: Izračunajte $\underbrace{\int_0^\infty \int_0^\infty e^{-(x+y)^2}\, dy\, dx}$

zapišemo tudi z

! $\int_0^\infty dx \int_0^\infty e^{-(x+y)^2}\, dy$!

$$\int_0^\infty \int_0^\infty e^{-(x+y)^2}\, dy\, dx = \int_0^\infty \Big(\underbrace{\int_0^\infty e^{-(x+y)^2}\, dy}_{} \Big)\, dx$$

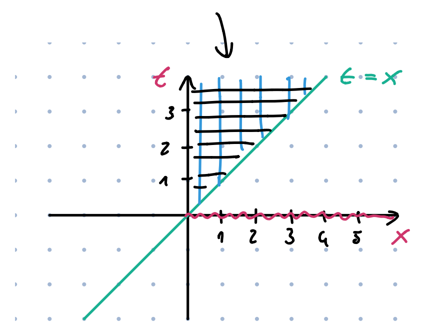
$t = x+y$ $\quad$ $y=0 \Rightarrow t=x$

$dt = dy$ $\quad$ $y=\infty \Rightarrow t=\infty$

$x \in [0,\infty) \Rightarrow t \in [x,\infty)$

$\Downarrow$

$t \in [0,\infty) \rightarrow x \in [0,t]$



$$= \int_0^\infty \Big(\int_x^\infty e^{-t^2}\, dt\Big)\, dx =$$

$$= \int_0^\infty dt \int_0^t e^{-t^2}\, dx =$$

$$= \int_0^\infty e^{-t^2} \cdot x \Big|_{x=0}^{t}\, dt = \int_0^\infty e^{-t^2} \cdot t\, dt =$$

$u = t^2$

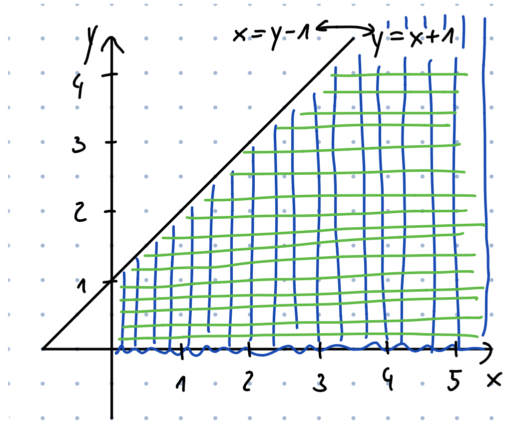
$du = 2t\,dt$

$$= \int_0^\infty e^{-u} \cdot \cancel{t} \cdot \frac{du}{2\cancel{t}} =$$

$$= \frac{1}{2}\, \frac{e^{-u}}{-1}\Big|_{u=0}^{\infty} = -\frac{1}{2}(0-1) = \underline{\underline{\frac{1}{2}}}$$

(1): Zamenjajte vrstni red integriranja $\int_0^\infty dx \int_0^{x+1} f(x,y)\,dy$.

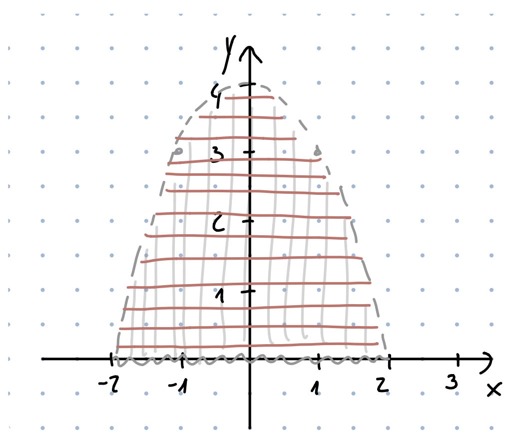
- $x \in [0, \infty) \rightarrow$ za fiksen x : $y \in [0, x+1]$



- $y \in [0, \infty) \longrightarrow$ za fiksen y :
  - če $y \in [0,1]$: $x \in [0, \infty)$
  - če $y \geq 1$: $x \in [y-1, \infty)$

$$\int_0^\infty dx \int_0^{x+1} f(x,y)\,dy = \int_0^\infty dy \left(\int_{\begin{cases} 0 ; & y \in [0,1] \\ y-1 ; & y \geq 1 \end{cases}}^{\infty} f(x,y)\,dx \right) = \int_0^1 dy \int_0^\infty f(x,y)\,dx + \int_1^\infty \int_{y-1}^\infty f(x,y)\,dx$$

(1): Zamenjajte vrstni red integriranja $\int_{-2}^{2} dx \int_0^{4-x^2} f(x,y)\,dy$.

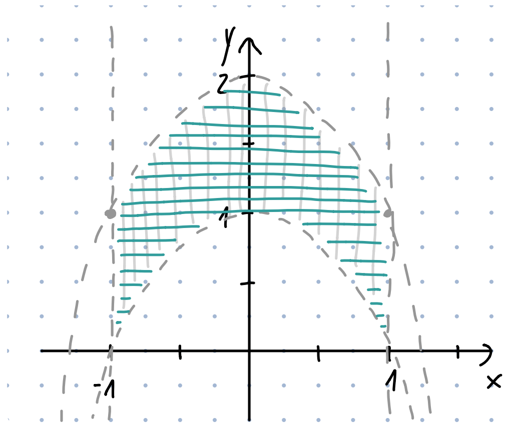


- $x \in [-2, 2] \longrightarrow y \in [0, 4-x^2]$
- $y \in [0, 4] \rightarrow x \in [-\sqrt{4-y}, \sqrt{4-y}]$

$$y = 4 - x^2$$
$$4 - y = x^2$$

$$\int_{-2}^{2} dx \int_0^{4-x^2} f(x,y)\,dy = \int_0^4 dy \int_{-\sqrt{4-y}}^{\sqrt{4-y}} f(x,y)\,dx$$

D: Zamenjajte vrstni red integriranja $\int\limits_{-1}^{1} dx \int\limits_{1-x^2}^{2-x^2} f(x,y)\,dy$.



- $x \in [-1,1] \rightarrow y \in [1-x^2, 2-x^2]$

- $y \in [0,2] \Rightarrow$
  - $y \leq 1: \; x \in [-1, -\sqrt{1-y}] \cup [\sqrt{1-y}, 1]$
  - $y \geq 1: \; x \in [-\sqrt{2-y}, \sqrt{2-y}]$

$1-x^2 = y \Rightarrow x^2 = 1-y$

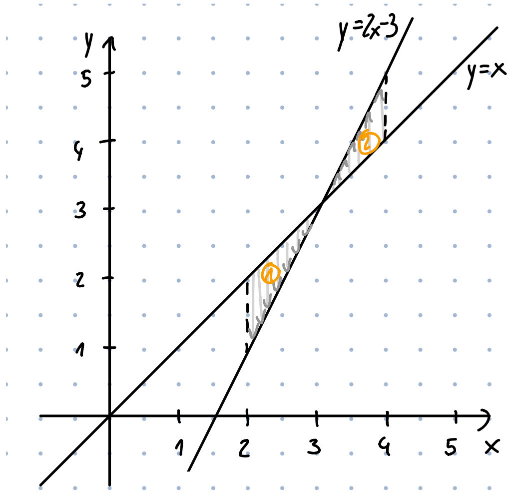
$2-x^2 = y \Rightarrow x^2 = 2-y$

$$\int\limits_{-1}^{1} dx \int\limits_{1-x^2}^{2-x^2} f(x,y)\,dy = \int\limits_{0}^{2} dy \int\limits_{\dots} f(x,y)\,dx$$

$$= \int\limits_{0}^{1} dy \left(\int\limits_{-1}^{-\sqrt{1-y}} f(x,y)\,dx + \int\limits_{\sqrt{1-y}}^{1} f(x,y)\,dx \right) + \int\limits_{1}^{2} dy \left(\int\limits_{-\sqrt{2-y}}^{\sqrt{2-y}} f(x,y)\,dx \right) =$$

$$= \int\limits_{0}^{1} dy \int\limits_{-1}^{-\sqrt{1-y}} f(x,y)\,dx + \int\limits_{0}^{1} dy \int\limits_{\sqrt{1-y}}^{1} f(x,y)\,dx + \int\limits_{1}^{2} dy \int\limits_{-\sqrt{2-y}}^{\sqrt{2-y}} f(x,y)\,dx$$

11*: Zamenjajte vrstni red integriranja $\int_2^4 dx \int_x^{2x-3} f(x,y)\,dy$.



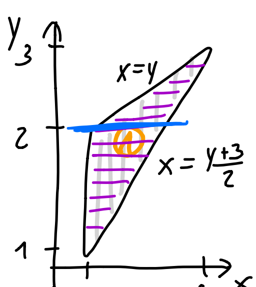
- $x \in [2,3] \rightarrow y \in [x, 2x-3]$

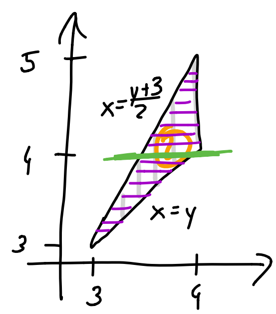
$y = 2x-3 \rightarrow x = \frac{y+3}{2}$

! $$\int_2^4 dx \int_x^{2x-3} f(x,y)\,dy = \int_2^3 dx \int_x^{2x-3} f(x,y)\,dx + \int_3^4 dx \int_x^{2x-3} f(x,y)\,dx =$$

$$= \underbrace{- \int_2^3 dx \int_{2x-3}^{x} f(x,y)\,dx}_{1} + \underbrace{\int_3^4 dx \int_x^{2x-3} f(x,y)\,dx}_{2} =$$

$$= -\left(\int_1^2 dy \int_2^{\frac{y+3}{2}} f(x,y)\,dx + \int_2^3 dy \int_y^{\frac{y+3}{2}} f(x,y)\,dx \right) + \int_3^4 dy \int_{\frac{y+3}{2}}^{x} f(x,y)\,dx + \int_4^5 dy \int_{\frac{y+3}{2}}^{4} f(x,y)\,dx$$





$$= \int_1^2 dy \int_{\frac{y+3}{2}}^{2} f(x,y)\,dx + \overbrace{\int_2^3 dy \int_{\frac{y+3}{2}}^{y} f(x,y)\,dx + \int_3^4 dy \int_{\frac{y+3}{2}}^{y} f(x,y)\,dx}^{= \int_2^4 dy \int_{\frac{y+3}{2}}^{y} f(x,y)\,dx} + \int_4^5 dy \int_{\frac{y+3}{2}}^{4} f(x,y)\,dx$$

UVEDBA NOVIH SPREMENLJIVK

Spomnimo se uvedbe nove spremenljivke za enojni integral:

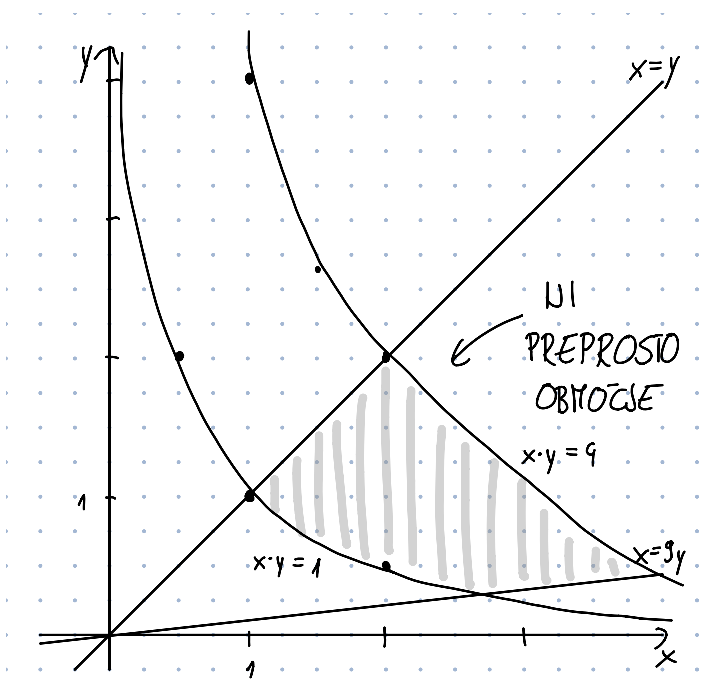
$$\int_a^b f(g(x))\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(t)\,dt \quad \text{za} \quad t=g(x) \text{ in } g \text{ injektivna.}$$

Dvojni integral: $$\iint_D f(x,y)\,dx\,dy = \iint_\Delta f(g(u,v),h(u,v))\cdot |J|\,du\,dv\,;$$

$$J=\begin{vmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v}\end{vmatrix}$$

N: Izračunajte $\iint_D x\,dx\,dy$

D območje v I. kvadrantu $(x>0, y>0)$ in je omejen s krivuljami $x=y$, $x=9y$, $x\cdot y=1$ in $x\cdot y=9$.



$\Rightarrow \quad x\geq y \Rightarrow \frac{x}{y}\geq 1$

$x\leq 9y \Rightarrow \frac{x}{y}\leq 9$

$$\boxed{1\leq \frac{x}{y}\leq 9}$$

$$\boxed{1\leq x\cdot y\leq 9}$$

$u=\frac{x}{y} \qquad u\in[1,9]$

$v=x\cdot y \qquad v\in[1,9]$

Želimo izrazit $x = g(u,v)$

$y = h(u,v)$

$u \cdot y = x$ $\quad \Rightarrow x = \sqrt{uv}$

$x \cdot y = v \Rightarrow u \cdot y \cdot y = v \Rightarrow y^2 = \frac{v}{u} \Rightarrow y = \sqrt{\frac{v}{u}}$

$g(u,v) =$

$h(u,v) = \sqrt{\frac{v}{u}}$

$$J = \begin{vmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2}\sqrt{v} \cdot \frac{1}{\sqrt{u}} & \frac{1}{2}\sqrt{u} \cdot \frac{1}{\sqrt{v}} \\ -\frac{1}{2}\sqrt{\frac{v}{u^3}} & \frac{1}{2}\frac{1}{\sqrt{u}} \cdot \frac{1}{\sqrt{v}} \end{vmatrix} = \frac{1}{4}\,\frac{1}{u} + \frac{1}{4} \cdot \frac{1}{u} = \frac{1}{2} \cdot \frac{1}{u} > 0$$

$$\iint_D x\,dx\,dy = \int_1^9 du \int_1^4 \sqrt{uv} \cdot \frac{1}{2} \cdot \frac{1}{u}\,dv = \frac{1}{2}\int_1^9 du \cdot \int_1^4 \frac{1}{\sqrt{u}} \cdot \sqrt{v}\,dv =$$

$$= \frac{1}{2}\int_1^9 \left(\frac{1}{\sqrt{u}} \cdot \frac{v^{\frac{3}{2}}}{\frac{3}{2}} \right) \Bigg|_{v=1}^{4} du = \frac{1}{\cancel{2}} \int_1^9 \frac{1}{\sqrt{u}} \cdot \frac{\cancel{2}}{3} \cdot \left(2^3 - 1\right) du =$$

$$= \frac{7}{3} \cdot \int_1^9 \frac{1}{\sqrt{u}}\,du = \frac{7}{3} \cdot \frac{\sqrt{u}}{\frac{1}{2}} \Bigg|_1^9 = \frac{7}{3} \cdot 2 \cdot (3-1) = \underline{\underline{\frac{28}{3}}}$$

NOVE SPREMENLJIVKE SMO UVEDLI ZARADI "GRDEGA" OBMOČJA.

15: Izračunajte integral $\displaystyle\iint\limits_{\substack{1<x^5y^2<2\\ x^3y>3}} \frac{1}{x^4y^2}\,dx\,dy.$

$$u = x^5y^2$$
$$v = x^3y$$

$$y = \frac{v}{x^3} \rightarrow u = x^5 \cdot \frac{v^2}{x^6} = \frac{v^2}{x}$$

$$x \cdot u = v^2 \qquad y = \frac{v}{x^3} = \frac{v}{v^6} \cdot u^3 = \frac{u^3}{v^5}$$

$$\boxed{x = \frac{v^2}{u}} \qquad \boxed{y = \frac{u^3}{v^5}}$$

$$g(u,v) = \frac{v^2}{u} \qquad h(u,v) = \frac{u^3}{v^5}$$

$$J = \begin{vmatrix} -\frac{v^2}{u^2} & 2\frac{v}{u} \\ \frac{3u^2}{v^5} & -5\frac{u^3}{v^6} \end{vmatrix} = 5\frac{v^2}{u^2} \cdot \frac{u^3}{v^6} - 6 \cdot \frac{u^2}{v^5} \cdot \frac{v}{u} = 5 \cdot \frac{u}{v^4} - 6 \cdot \frac{u}{v^4} = -\frac{u}{v^4}$$

$$\iint\limits_{\dots} \frac{1}{x^4y^2}\,dx\,dy = \int_1^2 du \int_3^\infty \frac{v^2}{u^2} \cdot \left|-\frac{u}{v^4}\right| dv = \int_1^2 du \int_3^\infty \frac{1}{u} \cdot \frac{1}{v^2}\,dv =$$

$1 < u < 2$

$v > 3$

$$= \int_1^2 \frac{1}{u} \cdot \left(-\frac{1}{v}\right)\Bigg|_{v=3}^{\infty} du =$$

$$\frac{1}{x^4y^2} = \frac{1}{\frac{\cancel{v^8}}{\cancel{u^4}} \cdot \frac{u^{\cancel{6}2}}{v^{\cancel{10}2}}} = \frac{v^2}{u^2}$$

$$= \int_1^2 \frac{1}{u}\left(-0 - \left(-\frac{1}{3}\right)\right) du =$$

$$= \int_1^2 \frac{1}{3} \cdot \frac{1}{u}\,du = \frac{1}{3} \ln|u| \Big|_{u=1}^{2} = \frac{1}{3}\ln 2$$

(N:) Izračunajte $\int_0^\infty \int_0^\infty \underbrace{(x-y)^2 e^{-(x+y)^2}}_{\text{KOMPLICIRANA FUNKCIJA}} \, dy\, dx$.

REŠITEV: NOVE SPREMENLJIVKE

$$u = x-y$$
$$v = x+y$$

$$x = \frac{u+v}{2} \qquad g(u,v) = \frac{u+v}{2}$$
$$y = \frac{v-u}{2} \qquad h(u,v) = \frac{v-u}{2}$$

$$\Rightarrow J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Območje: $x \in [0,\infty) \Rightarrow x \geq 0 \Rightarrow \frac{u+v}{2} \geq 0 \Rightarrow u+v \geq 0 \Rightarrow u \geq -v$

$y \in [0,\infty) \Rightarrow y \geq 0 \Rightarrow \frac{v-u}{2} \geq 0 \Rightarrow v-u \geq 0 \Rightarrow u \leq v$

$$\Uparrow$$
$$-v \leq u \leq v$$

$$v = \underset{\substack{\cup\!\shortmid \\ 0}}{x} + \underset{\substack{\cup\!\shortmid \\ 0}}{y} \in [0,\infty)$$

$$\int_0^\infty \int_0^\infty (x-y)^2 e^{-(x+y)^2} \, dy\, dx = \int_0^\infty dv \int_{-v}^{v} u^2 e^{-v^2} \cdot \left|\frac{1}{2}\right| du =$$

$$= \frac{1}{2}\int_0^\infty e^{-v^2} \cdot \frac{u^3}{3}\Big|_{u=-v}^{v} dv =$$

$$= \frac{1}{2}\int_0^\infty e^{-v^2} \cdot \left(\frac{v^3}{3} - \frac{(-v)^3}{3}\right) dv =$$

$$= \int_0^\infty e^{-v^2} \frac{1}{3} v^3 \, dv =$$

$t = v^2$

$dt = 2v\,dv$

$$= \int_0^\infty e^{-t} \cdot \frac{1}{3} \cdot t \cdot \cancel{v} \cdot \frac{1}{2\cancel{v}} \, dt =$$

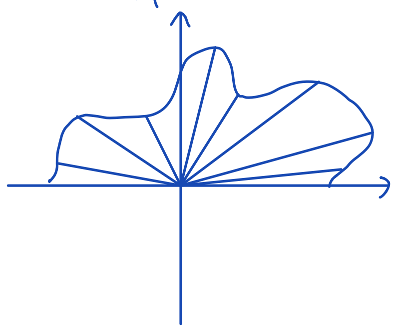
$$= \frac{1}{6} \cdot \int_0^\infty e^{-t} \cdot t \underset{\substack{\uparrow \\ \Gamma \text{ ali perpartes}}}{=} \frac{1}{6} \cdot \Gamma(2) = \underline{\underline{\frac{1}{6}}}$$

POLARNE KOORDINATE

$x = r\cdot\cos\varphi \qquad r > 0$

$y = r\cdot\sin\varphi \qquad 0 \le \varphi < 2\pi$

$$J = \begin{vmatrix} \cos\varphi & -r\cdot\sin\varphi \\ \sin\varphi & r\cdot\cos\varphi \end{vmatrix} = r\cdot\cos^2\varphi + r\cdot\sin^2\varphi = r$$

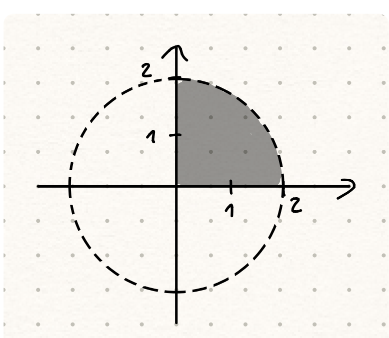


N: Izračunajte $\iint\limits_{\substack{x,y>0 \\ x^2+y^2<4}} \frac{xy}{(1+(x^2+y^2)^2)^2}\,dx\,dy$.

$x = r\cdot\cos\varphi$
$y = r\cdot\sin\varphi$

$$\frac{xy}{(1+(x^2+y^2)^2)^2} = \frac{r^2\cdot\cos\varphi\cdot\sin\varphi}{(1+(r^2\cos^2\varphi+r^2\sin^2\varphi)^2)^2} = \frac{r^2\cdot\cos\varphi\cdot\sin\varphi}{(1+r^4)^2} = \frac{r^2\cdot\frac{1}{2}\cdot\sin 2\varphi}{(1+r^4)^2}$$

Območje



$x^2+y^2<4 \Rightarrow r^2<4 \Rightarrow \underline{\underline{r<2}}$

$r\cdot\cos\varphi>0 \Rightarrow \cos\varphi>0 \Rightarrow \varphi\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
$r\cdot\sin\varphi>0 \Rightarrow \sin\varphi>0 \Rightarrow \varphi\in(0,\pi)$ $\Rightarrow \underline{\underline{\varphi\in\left(0,\frac{\pi}{2}\right)}}$

$$\iint\limits_{\substack{x,y>0\\x^2+y^2<4}} \frac{xy}{(1+(x^2+y^2)^2)^2}\,dx\,dy = \int_0^{\frac{\pi}{2}} d\varphi \int_0^2 \frac{r^2\cdot\frac{1}{2}\sin 2\varphi}{(1+r^4)^2}\, r\,dr =$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2\varphi\, d\varphi \underbrace{\int_0^2 \frac{r^3}{(1+r^4)^2}\,dr}_{\substack{\text{ker ta ni odvisen od } \varphi,\\ \text{bo konstanta, zato že}\\ \text{lahko računamo integral s } \varphi.}} =$$

$$= \frac{1}{2}\cdot\left(-\frac{\cos 2\varphi}{2}\right)\Bigg|_{\varphi=0}^{\frac{\pi}{2}} \cdot \int_0^2 \frac{r^3}{(1+r^4)^2}\,dr =$$

$t = 1+r^4$

$dt = 4\cdot r^3 dr$

$$= -\frac{1}{4}\cdot(\cos\pi - \cos 0)\cdot\int_1^{17} \frac{\cancel{r^3}}{t^2}\cdot\frac{dt}{4\cdot\cancel{r^3}} =$$

$$= -\frac{1}{4}\cdot(-1-1)\cdot\frac{1}{4}\cdot\frac{1}{t}\cdot(-1)\Bigg|_1^{17} =$$

$$= \frac{1}{8}\cdot(-1)\cdot\left(\frac{1}{17}-1\right) =$$

$$= -\frac{1}{8}\cdot\left(-\frac{16}{17}\right) =$$

$$= \underline{\underline{\frac{2}{17}}}$$

(N:) Izračunajte $\iint_D \sqrt{(x+1)^2+(y+1)^2}\,dxdy$. D... krog s središčem v izhodišču in polmerom $\sqrt{2}$

Premaknjene koordinate: $x = r\cdot\cos\varphi - 1$, $y = r\cdot\sin\varphi - 1$ ... $J = r$

$$\iint_D \sqrt{(x+1)^2+(y+1)^2}\,dxdy = \iint_\Delta \sqrt{r^2\cos^2\varphi + r^2\sin^2\varphi}\; r\,dr\,d\varphi =$$

$$= \iint_\Delta r^2\,dr\,d\varphi$$

D: $x^2+y^2 \leq 2$

$$r^2\cdot\cos^2\varphi - 2\cdot r\cdot\cos\varphi + \cancel{1} + r^2\cdot\sin^2\varphi - 2\cdot r\cdot\sin\varphi + \cancel{1} \leq \cancel{2}$$

$$r^2 - 2\cdot r\cdot(\cos\varphi + \sin\varphi) \leq 0$$

$$r \leq 2\cdot(\cos\varphi + \sin\varphi)$$

$$\cos\varphi + \sin\varphi \geq \frac{r}{2} \quad \ni \quad \cos\varphi + \sin\varphi \geq 0$$

$$\Updownarrow$$

$$\sin\left(\frac{\pi}{2} - \varphi\right) + \sin\varphi \geq 0$$

$$\Updownarrow \leftarrow \text{Faktorizacija}$$

$$2\cdot\sin\frac{\pi}{4}\cdot\cos\left(\frac{\pi}{4} - \varphi\right) \geq 0$$

$$\cos\left(\frac{\pi}{4} - \varphi\right) \geq 0$$

$$-\frac{\pi}{2} \leq \frac{\pi}{4} - \varphi \leq \frac{\pi}{2}$$

$$-\frac{\pi}{4} \leq \varphi \leq \frac{3\pi}{4} \quad !$$

$$\iint_D \sqrt{(x+1)^2+(y+1)^2}\,dx\,dy = \iint_\Delta r^2\,dr\,d\varphi =$$

$$= \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} d\varphi \int_0^{2\sin\varphi+\cos\varphi} r^2\,dr =$$

$$= \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(\frac{r^3}{3}\right)\Bigg|_{r=0}^{2\sin\varphi+\cos\varphi} d\varphi =$$

$$= \frac{1}{3}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} 8(\sin\varphi+\cos\varphi)^3\,d\varphi =$$

$$= \frac{8}{3}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(2\cdot\sin\frac{\pi}{4}\cdot\cos\left(\frac{\pi}{4}-\varphi\right)\right)^3 d\varphi =$$

$$= \frac{8}{3}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(\cancel{2}\cdot\frac{\sqrt{2}}{\cancel{2}}\cdot\cos\left(\frac{\pi}{4}-\varphi\right)\right)^3 d\varphi =$$

$$= \frac{8\cdot 2\sqrt{2}}{3}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos^3\left(\frac{\pi}{4}-\varphi\right) d\varphi =$$

$$= \frac{16\sqrt{2}}{3}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos^3\left(\varphi-\frac{\pi}{4}\right) d\varphi =$$

$$t = \sin\left(\varphi-\frac{\pi}{4}\right)$$
$$dt = \cos\left(\varphi-\frac{\pi}{4}\right)$$

$$= \frac{16\sqrt{2}}{3}\int_{\sin(-\frac{\pi}{2})}^{\sin(\frac{\pi}{2})} \left(1-\sin^2\left(\varphi-\frac{\pi}{4}\right)\right)\cdot\cancel{\cos\left(\varphi-\frac{\pi}{4}\right)}\cdot\frac{dt}{\cancel{\cos\left(\varphi-\frac{\pi}{4}\right)}} =$$

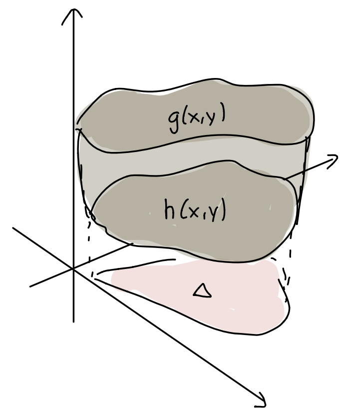
$$= \frac{16\sqrt{2}}{3}\int_{-1}^{1}(1-t^2)\,dt =$$

$$= \frac{16\sqrt{2}}{3}\cdot\left(t-\frac{t^3}{3}\right)\Bigg|_{t=-1}^{1} =$$

$$= \frac{16\sqrt{2}}{3}\cdot\left(\left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)\right) =$$

$$= \frac{16\sqrt{2}}{3}\cdot\left(\frac{2}{3}+\frac{2}{3}\right) = \frac{16\cdot 4\cdot\sqrt{2}}{9} = \underline{\underline{\frac{64\sqrt{2}}{9}}}$$

TROJNI INTEGRAL



D... območje v prostoru

D: $h(x,y) \leq z \leq g(x,y)$

$$\Rightarrow \iiint_D f(x,y,z)\,dxdydz = \iint_\Delta \left(\int_{h(x,y)}^{g(x,y)} f(x,y,z)\,dz \right) dxdy = \underline{\qquad\qquad}$$

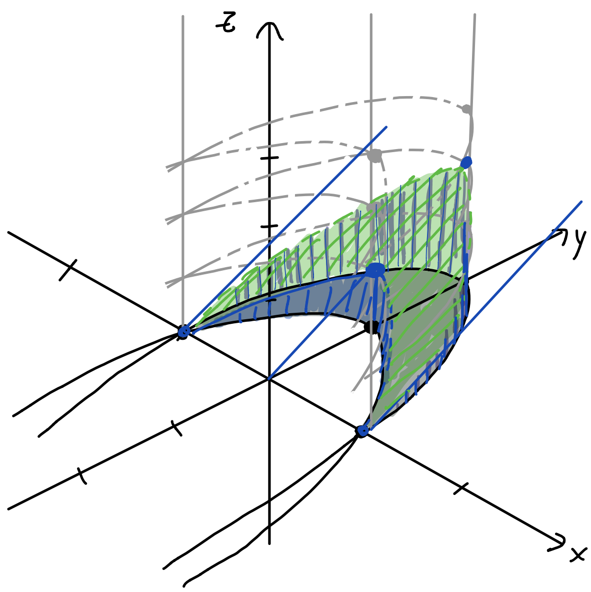
$\iint_\Delta$ prevedemo na dvakratni int.

$\iiint_D$ prevedemo na trikratni int.

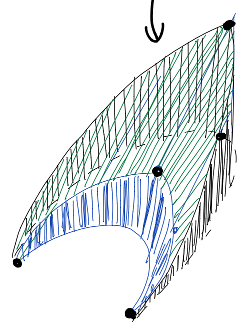
Volumen telesa D: $V(D) = \iiint_D 1\,dxdydz$

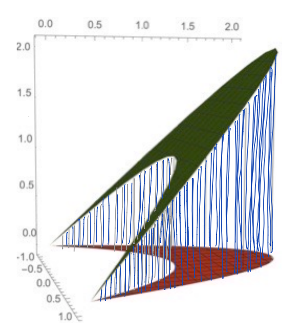
N: Izračunajte volumen telesa, ki ga omejujejo ploskve

$z=0,\ z=y,\ y=1-x^2,\ y=2-2x^2$



iščemo volumen tega telesa





$$V(D)=\iiint_D 1\,dx\,dy\,dz = \iint_{1-x^2\le y\le 2-2x^2}\left(\int_0^y 1\,dz\right)dx\,dy =$$

$$=\int_{-1}^{1}dx\int_{1-x^2}^{2-2x^2}dy\int_0^y 1\,dz =$$

$$=\int_{-1}^{1}dx\int_{1-x^2}^{2-2x^2} z\Big|_{z=0}^{y}\,dy =$$

$$=\int_{-1}^{1}dx\int_{1-x^2}^{2-2x^2} y\,dy =$$

$$=\int_{-1}^{1}\frac{y^2}{2}\Big|_{y=1-x^2}^{2-2x^2}dx =$$

$$=\frac{1}{2}\int_{-1}^{1}\left(4-8x^2+4x^4-1+2x^2-x^4\right)dx =$$

$$=\frac{1}{2}\int_{-1}^{1}\left(3-6x^2+3x^4\right)dx = \frac{3}{2}\int_{-1}^{1}\left(1-2x^2+x^4\right)dx =$$

$$=\frac{3}{2}\left(x-2\frac{x^3}{3}+\frac{x^5}{5}\right)\Big|_{x=-1}^{1} = \frac{3}{2}\left(1-\frac{2}{3}+\frac{1}{5}-(-1)+\left(-\frac{2}{3}\right)-\left(-\frac{1}{5}\right)\right)=$$

$$=\frac{3}{2}\left(2-\frac{4}{3}+\frac{2}{5}\right)=\frac{3}{2}\cdot\left(\frac{30-20+6}{15}\right)= \underline{\underline{\frac{8}{5}}}$$

(1): Izračunajte $\iiint\limits_{\substack{x<2 \\ x^2<y<x^3 \\ y<z<xy}} \frac{1}{z}\,dxdydz$.

Spodnja meja za x:

$$0 \le x^2 < y < x^3 \Rightarrow 0 < x^3 \Rightarrow x > 0$$

$$\text{in } x^2 < x^3 /:x^2 \Rightarrow 1 < x \Rightarrow x > 1$$

$$\iiint\limits_{\substack{x<2 \\ x^2<y<x^3 \\ y<z<xy}} \frac{1}{z}\,dxdydz = \int_1^2 dx \int_{x^2}^{x^3} dy \int_y^{xy} \frac{1}{z}\,dz =$$

$$= \int_1^2 dx \int_{x^2}^{x^3} \left(\ln|z| \Big|_{z=y}^{xy} \right) dy =$$

$$= \int_1^2 dx \int_{x^2}^{x^3} (\ln xy - \ln y)\, dy =$$

$$= \int_1^2 dx \int_{x^2}^{x^3} \ln \frac{x\cancel{y}}{\cancel{y}}\, dy =$$

$$= \int_1^2 \ln x \cdot y \Big|_{y=x^2}^{x^3} dx =$$

Per partes:

$u = \ln x \quad dv = x^3 - x^2$

$du = \frac{1}{x}dx \quad v = \frac{x^4}{4} - \frac{x^3}{3}$

$$= \int_1^2 \ln x \cdot (x^3 - x^2)\, dx =$$

$$= \left(\ln x \left(\frac{x^4}{4} - \frac{x^3}{3} \right) \Big|_1^2 - \int_1^2 \left(\frac{x^4}{4} - \frac{x^3}{3} \right) \cdot \frac{1}{x}\, dx \right) =$$

$$= \ln 2 \left(\frac{16}{4} - \frac{8}{3} \right) - \int_1^2 \left(\frac{x^3}{4} - \frac{x^2}{3} \right) dx \,) =$$

$$= \ln 2 \left(4 - \frac{8}{3} \right) - \left(\frac{x^4}{16} - \frac{x^3}{9} \right) \Big|_1^2 =$$

$$= \ln 2 \cdot \frac{12-8}{3} - \left(\frac{16}{16} - \frac{8}{9} - \frac{1}{16} + \frac{1}{9} \right) =$$

$$= \ln 2 \cdot \frac{4}{3} - \frac{15}{16} + \frac{7}{9} = \underline{\underline{\ln 2 \cdot \frac{4}{3} - \frac{23}{144}}}$$

(1): Izračunajte $\iiint\limits_V (x+y+z+1)^{-3}\,dx\,dy\,dz$, gdje je $V:=\{(x,y,z)\in\mathbb{R}^3;\ x,y,z\geq 0,\ x+y+z\leq 1\}$

$$x+y+z\leq 1 \Rightarrow 0\leq x\leq 1$$
$$\Rightarrow 0\leq y\leq 1-x$$
$$\Rightarrow 0\leq z\leq 1-x-y$$

$$\iiint\limits_V (x+y+z+1)^{-3}dx\,dy\,dz = \int_0^1 dx\int_0^{1-x}dy\int_0^{1-x-y}(x+y+z+1)^{-3}dz =$$

$$= \int_0^1 dx\int_0^{1-x}\frac{(x+y+z+1)^{-2}}{-2}\Bigg|_{z=0}^{1-x-y} dy =$$

$$= \int_0^1 dx\int_0^{1-x}\frac{(\cancel{x}+\cancel{y}+1-\cancel{x}-\cancel{y}+1)^{-2}-(x+y+1)^{-2}}{-2}dy =$$

$$= \int_0^1 dx\int_0^{1-x}\frac{2^{-2}-(x+y+1)^{-2}}{-2}dy =$$

$$= -\frac{1}{2}\int_0^1\left(2^{-2}y-\frac{(x+y+1)^{-1}}{-1}\right)\Bigg|_{y=0}^{1-x}dx =$$

$$= -\frac{1}{2}\int_0^1\left(\frac{1}{4}(1-x)+(\cancel{x}+1-\cancel{x}+1)^{-1}-(x+1)^{-1}\right)dx =$$

$$= -\frac{1}{8}\int_0^1(1-x)dx - \frac{1}{4}\int_0^1 1\,dx + \frac{1}{2}\int_0^1(x+1)^{-1}dx =$$

$$= -\frac{1}{8}\,\frac{(1-x)^2}{-2}\Bigg|_{x=0}^{1} - \frac{1}{4}\cdot 1 + \frac{1}{2}\ln|x+1|\Bigg|_{x=0}^{1} =$$

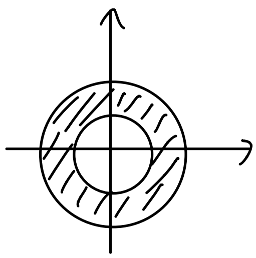
$$= \frac{1}{16}(0-1)-\frac{1}{4}+\frac{1}{2}\ln 2-\frac{1}{2}\cdot 0 =$$

$$= \underline{\underline{-\frac{5}{16}+\frac{1}{2}\ln 2}}$$

UVEDBA NOVIH SPREMENLJIVK

$$\boxed{\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_\Delta f(g(u,v,w),h(u,v,w),k(u,v,w))\cdot|J|\,du\,dv\,dw}$$

Z: Izračunajte prostornino telesa $T=\{(x,y,z)\in\mathbb{R}^3;\ 1\le x^2+y^2\le 4,\ 0\le z\le \frac{4}{x^2+y^2}\}$



Cilindrične koordinate:

$x = r\cdot\cos\varphi$

$y = r\cdot\sin\varphi$

$z = z$

$J = r$

$1\le x^2+y^2\le 4 \qquad 0\le z\le \frac{4}{x^2+y^2}$

$1\le r^2\le 4 \qquad 0\le z\le \frac{4}{r^2}$

$1\le r\le 2$

$\varphi\in[0,2\pi]$

$$V(T)=\iiint_T 1\,dx\,dy\,dz = \int_0^{2\pi}d\varphi\int_1^2 dr\int_0^{\frac{4}{r^2}} 1\cdot \overset{J}{r}\,dz = \int_0^{2\pi}d\varphi\int_1^2 \cancel{r}\cdot\frac{4}{r^2}\,dr =$$

$$=\int_0^{2\pi}d\varphi\int_1^2\frac{4}{r}\,dr = 4\int_0^{2\pi}\ln|r|\Big|_{r=1}^{2}d\varphi = 4\int_0^{2\pi}\ln 2-0\,d\varphi = 4\cdot\ln2\cdot2\pi=\underline{\underline{8\pi\ln 2}}$$

Z: Izračunajte prostornino telesa $T=\{(x,y,z)\in\mathbb{R}^3;\ x^2+y^2\le 2x,\ 0\le z\le xy,\ y\ge 0\}$

$x^2+y^2\le 2x$

$x^2-2x+y^2\le 0$

$x^2-2x+1-1+y^2\le 0$

$(x-1)^2+y^2\le 1 \ \longrightarrow\ r\le 1$

$$V(T)=\iiint_T 1\,dx\,dy\,dz = \int_0^{\pi}d\varphi\int_0^1 dr\int_0^{\frac12 r^2\sin2\varphi} r\,dz = \dots$$

$$= DN = \underline{\underline{\frac{2}{3}}}$$

Premaknjene cilindrične koordinate:

$x = r\cdot\cos\varphi + 1$

$y = r\cdot\sin\varphi$

$z = z$

$J = r$

$y\ge 0 \Rightarrow \sin\varphi\ge 0\Rightarrow \varphi\in[0,\pi]$

$0\le z\le r^2\cdot\underline{\sin\varphi\cdot\cos\varphi = 1^2\cdot\frac12\cdot\sin2\varphi}$

D: Izračunajte prostornino telesa

$$T=\{(x,y,z)\in\mathbb{R}^3 ;\ 1\le x^2+y^2+z^2\le 16,\ x^2+y^2\le z^2,\ z\ge 0\}$$

Sferične koordinate:

$x = r\cdot\cos\varphi\cdot\cos\vartheta$

$y = r\cdot\sin\varphi\cdot\cos\vartheta$

$z = r\cdot\sin\vartheta$

$J = r^2\cdot\cos\vartheta$

$$1\le x^2+y^2+z^2\le 16$$
$$1\le r^2\cos^2\varphi\cos^2\vartheta + r^2\sin^2\varphi\cos^2\vartheta + r^2\sin^2\vartheta\le 16$$
$$1\le r^2\cos^2\vartheta + r^2\sin^2\vartheta\le 16$$
$$1\le r^2\le 16$$
$$\underline{1\le r\le 4} \qquad \underline{\varphi\in[0,2\pi]}$$

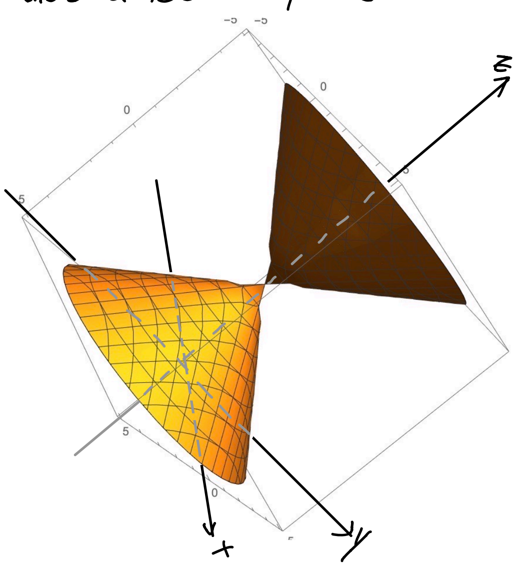
$$x^2+y^2\le z^2$$
$$r^2\cdot\cos^2\varphi\cdot\cos^2\vartheta + r^2\cdot\sin^2\varphi\cdot\cos^2\vartheta\le r^2\cdot\sin^2\vartheta$$
$$\cancel{r^2}\cdot\cos^2\vartheta\le\cancel{r^2}\cdot\sin^2\vartheta$$
$$\cos^2\vartheta\le\sin^2\vartheta$$

$* \ \vartheta\in[-\frac{\pi}{2},\frac{\pi}{2}]\Rightarrow\cos\vartheta\ge 0$

$z\ge 0\Rightarrow\sin\vartheta\ge 0$

$$\cos\vartheta\le\sin\vartheta$$
$$\tan\vartheta\ge 1$$
$$\underline{\vartheta\in[\frac{\pi}{4},\frac{\pi}{2}]}$$

Množica točk $x^2+y^2\le z^2$:



$$\iiint_T 1\,dx\,dy\,dz = \int_0^{2\pi}d\varphi\int_1^4 dr\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} r^2\cdot\cos\vartheta\,d\vartheta$$
$$= \int_0^{2\pi}d\varphi\int_1^4 r^2\cdot\sin\vartheta\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}dr =$$
$$= \int_0^{2\pi}d\varphi\int_1^4 r^2\cdot\left(1-\frac{\sqrt2}{2}\right)dr =$$
$$= \left(1-\frac{\sqrt2}{2}\right)\int_0^{2\pi}\frac{r^3}{3}\Big|_{r=1}^{4}d\varphi =$$
$$= \left(1-\frac{\sqrt2}{2}\right)\cdot\left(\frac{64}{3}-\frac{1}{3}\right)\cdot 2\pi = \underline{\underline{(2-\sqrt2)\pi\cdot 21}}$$

DN: Reši še primer za T brez pogoja $z\ge 0$!

R: $42\pi(2-\sqrt2)$