$\mathcal{A}(1,1) = (1,1) \Rightarrow b_{11} = 1,\quad b_{21} = 0$

$\mathcal{A}(-1,1) = (1,-1) \Rightarrow b_{12} = 0 \quad b_{22} = -1$

$\uparrow \; = -v_2$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Zapiši preslikavo P, ki preslika $\begin{matrix} e_1 \to v_1 \\ e_2 \to v_2 \end{matrix}$

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Kakšna je zveza med A, B, P?

Kolokvij: petek 18.12. ob 8^{00}.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\quad P = \begin{array}{cc} v_1 & v_2 \\ \end{array} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},\quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

B: v bazi $\{v_1, v_2\}$

PREHOD NA NOVO BAZO

$$\begin{tikzcd}
\mathbb{R}^2, \text{std. baza} \arrow[d, "A"'] \arrow[r, shift right, "P^{-1}"'] & \mathbb{R}^2, \{v_1, v_2\} \arrow[l, shift right, "P"'] \arrow[d, "B"] \\
\mathbb{R}^2, \text{std baza} & \mathbb{R}^2, \{v_1, v_2\} \arrow[l, "P"]
\end{tikzcd}$$

$A = P \, B \, P^{-1}$

$P: e_1 \to v_1$

P: 1. bazni vektor v bazi $\{v_1, v_2\}$ v vektor $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ v bazi $\{e_1, e_2\}$

2. $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ —"—

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} =$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

Lastne vrednosti in lastni vektorji

U: lin. psl., ki je v std. bazi pripada matrika A.
Iščemo take vektorje v, da velja

$$Av = \lambda v \quad \text{za neki } \lambda \,; \quad v \neq 0.$$

To pomeni, da je v v jedru matrike $A - \lambda I$:

$$Av = \lambda \cdot I v \Rightarrow (A - \lambda I)v = 0.$$

Vemo: Če je jedro netrivialno, matrika nima polnega ranga in je njena determinanta 0.

Torej: iščemo vse λ: $\det(A - \lambda I) = 0$. Ko te λ dobimo, poiščemo še pripadajoče v.

Primeri: Izračunaj lastne vrednosti in lastne vektorje matrike A.

$$A = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix} \qquad \det A - \lambda I = \det \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} =$$

$$= \begin{vmatrix} 2-\lambda & 4 \\ 5 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) - 20 =$$

karakteristični polinom

$$= \lambda^2 - 5\lambda + 6 - 20 = \lambda^2 - 5\lambda - 14 = (\lambda - 7)(\lambda + 2)$$

$\lambda_1 = -2$: v_1 je v jedru $A - (-2)I = \begin{bmatrix} 4 & 4 \\ 5 & 5 \end{bmatrix}$: rang 1

lastna vrednost

lastni vektor $\rightarrow v_1 = (-1, 1)$ $\quad (\lambda_1, v_1)$: lastni par

$\lambda_2 = 7$: $\begin{bmatrix} -5 & 4 \\ 5 & -4 \end{bmatrix}$ $\quad v_2 = (4, 5)$

V bazi $\{v_1, v_2\}$ preslikavi $\mathcal{A}$ pripada matrika

$$D = \begin{bmatrix} -2 & 0 \\ 0 & 7 \end{bmatrix} \qquad \begin{aligned} \mathcal{A}(v_1) &= -2 v_1 \\ \mathcal{A}(v_2) &= 7 v_2 \end{aligned}$$

$$P = \begin{bmatrix} -1 & 4 \\ 1 & 5 \end{bmatrix} \qquad A = PDP^{-1}$$

Izračunaj lastne vrednosti in lastne vektorje matrike

$$\det \begin{bmatrix} 3-\lambda & 2 & 2 \\ 1 & 4-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{bmatrix} = (3-\lambda)\Big((4-\lambda)(-1-\lambda)+4\Big) -$$

$$-2\,\big(1(-1-\lambda)+2\big) + 2\cdot\big(-4+2(4-\lambda)\big) =$$

$$-\lambda+3$$

$$= (3-\lambda)\cdot(\lambda^2-3\lambda) + 2\cdot(\lambda-1-4+8-2\lambda) =$$

$$= (3-\lambda)\cdot(\lambda^2-3\lambda) - 2(\lambda-3) =$$

$$= (3-\lambda)(\lambda^2-3\lambda+2) = (3-\lambda)(\lambda-1)(\lambda-2)$$

$$\lambda_1 = 1: \quad \begin{bmatrix} 2 & 2 & 2 \\ 1 & 3 & 1 \\ -2 & -4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\lambda_2 = 2: \quad \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \\ -2 & -4 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_3 = 3: \quad \begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & 1 \\ -2 & -4 & -4 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Kakšna je matrika v lastni bazi $\{v_1, v_2, v_3\}$?

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

Izračunaj lastne vrednosti in l. vektorje

$$\det \begin{bmatrix} -1-\lambda & 2i \\ -2i & 2-\lambda \end{bmatrix} = (-1-\lambda)(2-\lambda) - 2i(-2i) =$$
$$= \lambda^2 - \lambda - 2 - 4 = \lambda^2 - \lambda - 6 = (\lambda-3)(\lambda+2)$$

$$\lambda_1 = -2: \quad \begin{bmatrix} 1 & 2i \\ -2i & 4 \end{bmatrix} \qquad v_1 = \begin{bmatrix} -2i \\ 1 \end{bmatrix}$$

$$\lambda_2 = 3 \quad \begin{bmatrix} -4 & 2i \\ -2i & -1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} -1 \\ 2i \end{bmatrix}$$

$$D = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}, \quad P = \begin{bmatrix} -2i & -1 \\ 1 & 2i \end{bmatrix}$$

Pozor: Če je matrika realna in ima karakteristični polinom kompl. ničle, so vedno v konj. parih in l. vektorji so konjugirani! $\lambda_1 = \overline{\lambda_2} \Rightarrow v_1 = \overline{v_2}$

Če matrika NI REALNA, to ne drži.

Izračunaj l. vrednosti in l. vektorje za matriko

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \qquad \det(A-\lambda I) = (3-\lambda)^2 - 1 = 0$$

$(\lambda-3)^2 = 1$ oz. $\lambda - 3 = \pm 1$; $\lambda_1 = 4$, $\lambda_2 = 2$

$$\lambda_1 = 4 : \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} : \quad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 2 : \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad v_2 = -\begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = R(\pi/4)$$

Vektorja v_1, v_2 sta PRAVOKOTNA.

Pomembna družina matrik so ORTOGONALNE matrike, to so take, katerih vrstice (stolpci) predstavljajo ortonormirano bazo.
to so matrike Q: $Q \cdot Q^T = I$; $Q^{-1} = Q^T$

A simetrična lahko izberemo P ortogonalno.

Izračunaj A^{2020}

$$A = PDP^{-1}; \quad A^2 = PD\cancel{P^{-1}}\cancel{P}DP^{-1} = PD^2P^{-1}$$

$$A^n = PD^nP^{-1}$$

$$p(x) = \underset{\substack{\uparrow \\ A}}{a_n x^n} + \dots + \underset{\substack{\uparrow \\ I}}{a_0 \cdot 1}; \quad p(A) = P \cdot p(D) \cdot P^{-1}$$

$$p(x) = x^2 + 1 \rightarrow p(A) = A^2 + I = PD^2P^{-1} + PP^{-1} = P(D^2+I)P^{-1}$$

$$f(x)=\sum_{0}^{\infty} a_n x^n \quad \rightsquigarrow \quad f(A)=P\cdot \underbrace{\sum_{0}^{\infty} a_n D^n}_{= \begin{bmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_n) \end{bmatrix}} P^{-1}$$

kvadratne forme v $\mathbb{R}^2$:

$$f(x,y) = \underbrace{ax^2 + 2bxy + cy^2}\ : \qquad A = \begin{bmatrix} & \\ & \end{bmatrix}$$

Ideja: $\forall$ kvadratno formo je mogoče zapisati v obliki $\left\langle A\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle$ za A simetrično.

Potem o vedenju f odloči D, ki pripada A.

$$\underbrace{[x\ ,y]}_{= v^T} \underbrace{\begin{bmatrix} a & b \\ b & c \end{bmatrix}}_{A = QDQ^T} \overbrace{\begin{bmatrix} x \\ y \end{bmatrix}}^{=v} = [x \ \ y] \begin{bmatrix} ax+by \\ bx+cy \end{bmatrix} = ax^2 + xby + ybx + cy^2 = ax^2 + 2bxy + cy^2$$

$$v^T A v = \underbrace{v^T Q}_{(Q^T v)^T} D \underbrace{Q^T v}_{w} = w^T D w \qquad w = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

↳ tej formi v drugi bazi pripada $\alpha^2 d_1 + \beta^2 d_2$

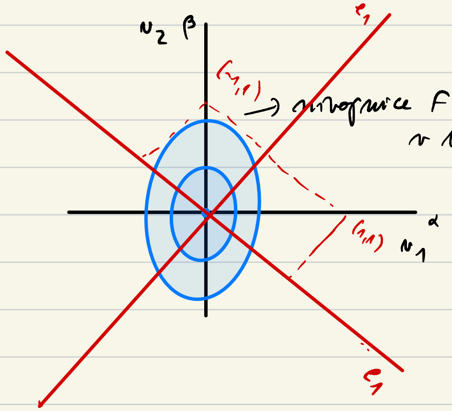
Primer: dana je forma $f(x,y) = 3x^2 + 2xy + 3y^2$. Pripada ji matrika

$$A = \begin{array}{c} \\ x \\ y \end{array}\begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \end{array} \quad ; \quad D = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

$\{v_1, v_2\}$: $\rightsquigarrow \; 4\alpha^2 + 2\beta^2 = c$: družina elips

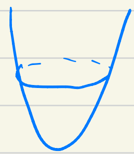
$\left(\frac{\alpha}{a}\right)^2 + \left(\frac{\beta}{b}\right)^2 = 1$: a, b: polosi

$\left(\frac{\alpha}{1/2}\right)^2 + \left(\frac{\beta}{1/\sqrt{2}}\right)^2 = c$



$\rightarrow$ nivojnice F v bazi $\{v_1, v_2\}$

MIN v 0



graf f: eliptični paraboloid

Kolokvij: jedro, slika (baza, dimenzija), zapis matrike za dano lin. psr., inverzna matrika.

Izračunaj l. vrednosti in l. vektorje naslednjih matrik:

$$A_1 = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 2 & -1 & 0 \\ 1 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$