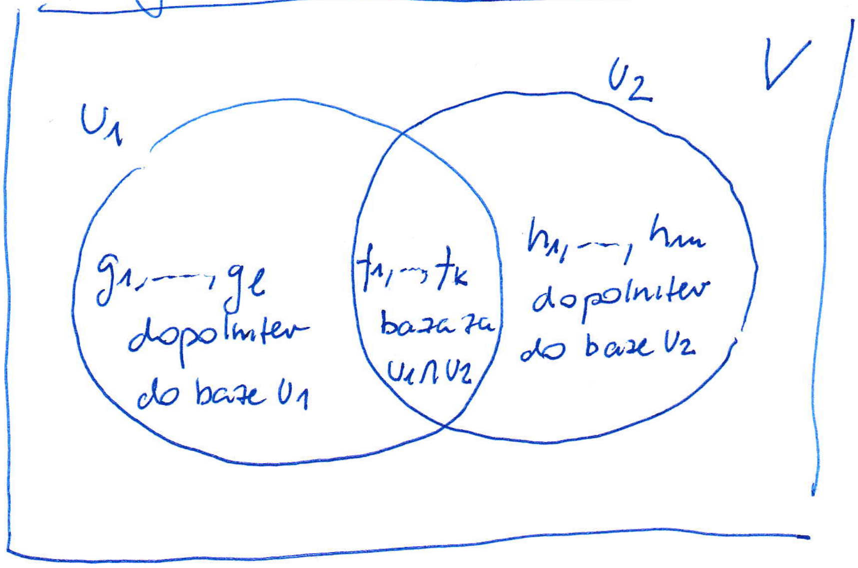
Dimenzijska formula za podprostore



$f_1, \dots, f_k, g_1, \dots, g_l$ so baza za U_1

$f_1, \dots, f_k, h_1, \dots, h_m$ so baza za U_2

$$\left.\begin{aligned} \dim(U_1 \cap U_2) &= k \\ \dim U_1 &= k+l \\ \dim U_2 &= k+m \end{aligned}\right\} \Rightarrow \begin{aligned} &\dim U_1 + \dim U_2 - \dim(U_1 \cap U_2) \\ &= (k+l) + (k+m) - k \\ &= k+l+m \end{aligned}$$

Dokazati moramo, da je $\dim(U_1 + U_2) = k+l+m$.

Zadošča konstruirati bazo s $k+l+m$ elementi.

Kandidat za bazo za U_1+U_2 so $f_1, \dots, f_k, g_1, \dots, g_l, h_1, \dots, h_m$

Dokazati moramo, da so ti vektorji ogrodje in da so linearno neodvisni

•) Zakaj so ti vektorji ogrodje?

Vzamemo poljuben $u \in U_1 + U_2$. Po definiciji vsote podprostorov je $u = u_1 + u_2$ za nek $u_1 \in U_1$ in $u_2 \in U_2$

Ker so $f_1, \dots, f_k, g_1, \dots, g_l$ ogrodje za U_1,

in ker je $u_1 \in U_1$, lahko razvijemo

$$u_1 = \alpha_1 f_1 + \dots + \alpha_k f_k + \beta_1 g_1 + \dots + \beta_l g_l$$

Ker so $f_1, \dots, f_k, h_1, \dots, h_m$ ogrodje za U_2

in ker je $u_2 \in U_2$, lahko razvijemo

$$u_2 = \gamma_1 f_1 + \dots + \gamma_k f_k + \delta_1 h_1 + \dots + \delta_m h_m$$

Odtod sledi, da je

$$u = u_1 + u_2 = (\alpha_1 + \gamma_1) f_1 + \dots + (\alpha_k + \gamma_k) f_k$$
$$+ \beta_1 g_1 + \dots + \beta_l g_l + \delta_1 h_1 + \dots + \delta_m h_m$$
$$\in \mathrm{Lin}\{f_1, \dots, f_k, g_1, \dots, g_l, h_1, \dots, h_m\}$$

Dokazali smo, da je $\mathrm{Lin}\{f_i, g_j, h_k\} = U_1 + U_2$

•) Zakaj so $f_1, \dots, f_k, g_1, \dots, g_l, h_1, \dots, h_m$

linearno neodvisni?

Recimo, da je

$$(*) \quad \alpha_1 f_1 + \dots + \alpha_k f_k + \beta_1 g_1 + \dots + \beta_l g_l + \gamma_1 h_1 + \dots + \gamma_m h_m = 0$$

Radi bi pokazali, da je $\alpha_1 = \dots = \alpha_k = 0$, $\beta_1 = \dots = \beta_l = 0$

in $\gamma_1 = \dots = \gamma_m = 0$

Odtod sledi, da je

$$\underbrace{\alpha_1 f_1 + \dots + \alpha_k f_k + \beta_1 g_1 + \dots + \beta_l g_l}_{\in U_1} = \underbrace{(-\gamma_1) h_1 + \dots + (-\gamma_m) h_m}_{\in U_2}$$

Torej sta obe strani v $U_1 \cap U_2$.

Ker ima $U_1 \cap U_2$ bazo $f_1, \dots, f_k$, lahko razvijemo desno stran po tej bazi

$$(-\gamma_1) h_1 + \dots + (-\gamma_m) h_m = \delta_1 f_1 + \dots + \delta_k f_k$$

Če damo vse na isto stran, dobimo

$$\gamma_1 h_1 + \dots + \gamma_m h_m + \delta_1 f_1 + \dots + \delta_k f_k = 0$$

Upoštevamo, da ~~je~~ so $f_1, \dots, f_k, h_1, \dots, h_m$ linearno neodvisni (ker so baza za U_2)

Dobimo $\gamma_1 = \dots = \gamma_m = 0$ in $\delta_1 = \dots = \delta_k = 0$

Če to upoštevamo v (*), dobimo

$$\alpha_1 f_1 + \dots + \alpha_k f_k + \beta_1 g_1 + \dots + \beta_l g_l = 0$$

Upoštevamo, da so $f_1, \dots, f_k, g_1, \dots, g_l$ lin. neodvisni (ker so baza za U_1). Dobimo

$\alpha_1 = \dots = \alpha_k = 0$ in $\beta_1 = \dots = \beta_l = 0$

$$P_{B \leftarrow B} = ?$$

$$u_1 = 1 \cdot u_1 + 0 \cdot u_2 + \dots + 0 \cdot u_n$$

$$u_2 = 0 \cdot u_1 + 1 \cdot u_2 + \dots + 0 \cdot u_n$$

$$\vdots$$

$$u_n = 0 \cdot u_1 + 0 \cdot u_2 + \dots + 1 \cdot u_n$$

$$? = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ \vdots & & & \vdots \\ 0 & 0 & & 1 \end{bmatrix} = I$$

Torej je $P_{B \leftarrow B} = I$