LINEARNA ALGEBRA 2020/21
14. VAJE: 4. 1. 2021

1. Poišči vse baze $\mathbb{Z}_2$-vektorskega prostora $\mathbb{Z}_2^3$ in $\mathbb{Z}_3$-vektorskega prostora $\mathbb{Z}_3^3$.

2. Ali lahko abelovo grupo $\mathbb{Z}_4$ opremimo s skalarnim množenjem $\mathbb{Z}_2 \times \mathbb{Z}_4 \to Z_4$, da bo $\mathbb{Z}_4$ $\mathbb{Z}_2$-vektorski prostor?

3. Naj bodo $x = (-1, 2, 3)^T, y = (3, 4, 2)^T, z = (2, 6, 6)^T$ in $w = (-9, -2, 5)^T$. Ali sta $z, w \in \mathrm{Lin}\{x, y\} \subset \mathbb{R}^3$?

4. Zapiši bazo vektorskega prostora rešitev enačbe

$$\begin{bmatrix} 4 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

   nad $\mathbb{Z}_5$ in $\mathbb{R}$.

5. Za vektorska podprostora $U = \mathrm{Lin}\{(1, 1, 1, 1, 1), (1, 0, 0, 0, -1)\}$ in $V = \mathrm{Lin}\{(-1, -1, 0, 0, 1), (3, 2, 2, 2, 1), (2, 1, 2, 2, 2)\}$ določi bazo preseka in vsote.

6. Za vektorska podprostora U, V vektorskega prostora W dokaži ekvivalence

   (i) $U + V = V \iff U \subseteq V$

   (ii) $U \cap V = V \iff V \subseteq U$

   (iii) $U + V = U \cup V \iff (U \subseteq V$ ali $V \subseteq U)$.

   Pokaži, da je $U \cup V$ vektorski podprostor natanko tedaj, ko je $(U \subseteq V$ ali $V \subseteq U)$

7. Poišči bazo podprostora $\{p \in \mathbb{R}_n[x] \mid p(0) = p(1) = p(2)\} \subset \mathbb{R}_n[x]$. (In bazo kakšnega komplementa.)

1. Poišči vse baze $\mathbb{Z}_2$-vektorskega prostora $\mathbb{Z}_2^3$ in [redacted]

$$\mathbb{Z}_2^3 = \left\{ \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$$

- $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$
- $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}$
- $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}$
- $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}$
- $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}$
- $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}$
- $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}$ ...

2. Ali lahko abelovo grupo $\mathbb{Z}_4$ opremimo s skalarnim množenjem $\mathbb{Z}_2 \times \mathbb{Z}_4 \to \mathbb{Z}_4$, da bo $\mathbb{Z}_4$ $\mathbb{Z}_2$-vektorski prostor?

$(\mathbb{Z}_4, +)$ $\qquad \mathbb{Z}_2 = \{0, 1\}$

$\mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$

Zakaj to ne gre?

$1 \cdot v = v \qquad 1 \in \mathbb{Z}_2 \quad v \in \mathbb{Z}_4$

$(1+1) \equiv 0 \mod 2 \qquad 0 = \bar{1} + \bar{1} = \bar{2} \not\to$

$0 = 0 \cdot v = (1+1) \cdot v = v + v$

$\mathbb{Z}_4$ ni v.p. na nobenem polju!

$\forall v \in \mathbb{Z}_4 \quad 4 \cdot v = v+v+v+v = 0$

1) $1+1+1+1 = 0 \in F$

$2 \cdot 2 = 4 = 0 \quad$ 2 ni obrnljiva v F

$2 = 0 \in F \rightsquigarrow$ Nadaljujemo kot zgoraj $\rightarrow\!\!\times$

2) 4 je obrnljiva v F

$\forall v \quad 0 = 4 \cdot v \quad / \cdot 4^{-1}$

$0 = v$

$\rightarrow\!\!\times$

3. Naj bodo $x = (-1, 2, 3)^T, y = (3, 4, 2)^T, z = (2, 6, 6)^T$ in $w = (-9, -2, 5)^T$. Ali sta $z, w \in \text{Lin}\{x, y\} \subset \mathbb{R}^3$?

$$\text{Lin}\{v_1, v_2, \dots, v_n\} = \left\{ \sum_{i=1}^{n} \alpha_i v_i \;\middle|\; \alpha_i \in \mathbb{R} \right\}$$

Zanima nas ali obstajata $\alpha_1, \beta_1 \in \mathbb{R}$ in α_2, β_2 da bo

$$\alpha_1 x + \beta_1 y = z$$

$$\alpha_2 x + \beta_2 y = w$$

$\Updownarrow$ matrično

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{bmatrix} = \begin{bmatrix} z & w \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} -1 & 3 & 2 & -9 \\ 2 & 4 & 6 & -2 \\ 3 & 2 & 6 & 5 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc} -1 & 3 & 2 & -9 \\ 0 & 10 & 10 & -20 \\ 0 & 11 & 12 & -22 \end{array}\right] \rightsquigarrow$$

$$\rightsquigarrow \left[\begin{array}{cc|cc} -1 & 3 & 2 & -9 \\ 0 & 10 & 10 & -20 \\ 0 & 1 & 2 & -2 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc} -1 & 3 & 2 & -9 \\ 0 & 0 & -10 & 0 \\ 0 & 1 & 2 & -2 \end{array}\right] \rightsquigarrow$$

$\rightsquigarrow z \notin \text{Lin}\{x, y\}$

$$\left[\begin{array}{cc|c} -1 & 3 & -9 \\ 0 & 1 & -2 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right] \rightsquigarrow w = 3x - 2y$$

$w \in \text{Lin}\{x, y\}$

Alternativno:

Preverimo, da sta x, y lin. neodvisna. Preverimo $\det[x,y,z] \neq 0$

$\det[x,y,z] = 0$

4. Zapiši bazo vektorskega prostora rešitev enačbe

$$\begin{bmatrix} 4 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

nad $\mathbb{Z}_5$ in $\mathbb{R}$.

$\{x \in F^n \mid Ax = 0\} \subseteq F^n$

$\begin{bmatrix} 4 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 4 & 3 \end{bmatrix} \in M_3(\mathbb{R})$ $\qquad$ $\begin{bmatrix} 4 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 4 & 3 \end{bmatrix} \in M_3(\mathbb{Z}_5)$

Dve popolnoma različni matriki

Nad $\mathbb{R}$

$$\begin{bmatrix} 4 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 4 & 3 \end{bmatrix} \overset{-1}{\rightsquigarrow} \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 3 \end{bmatrix} \overset{-2,\,-3}{\rightsquigarrow} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 5 & 0 \\ 0 & 10 & 3 \end{bmatrix} \rightsquigarrow$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \rightsquigarrow \text{Prostor rešitev } \{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \}$$

ima bazo $B = \{\} = \emptyset$

Nad $\mathbb{Z}_5$

$-2 \equiv 3 \bmod 5$ $\quad 5 \equiv 0 \bmod 5 \quad 10 \equiv 0 \bmod 5$

$$\begin{bmatrix} 4 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 4 & 3 \end{bmatrix} \xrightarrow{-1} \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$1-2\cdot 3 = -5 \equiv 0 \bmod 5$

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$x_1 + 3x_2 = 0 \qquad x_1 = -3x_2 = 2x_2$

$x_3 = 0$

rešitve $\to \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} t \;\middle|\; t \in \mathbb{Z}_5 \right\}$

Možne baze

$B_1 = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \right\}.$

5. Za vektorska podprostora $U = \mathrm{Lin}\{(1,1,1,1,1),(1,0,0,0,-1)\}$ in $V = \mathrm{Lin}\{(-1,-1,0,0,1),(3,2,2,2,1),(2,1,2,2,2)\}$ določi bazo preseka in vsote.

$V, U \subseteq \mathbb{R}^5$

$U+V$ – najmanjši v.p., ki vsebuje U in V.

$\| $

$\{u+v \mid u \in U, v \in V\}$

$\|$

$\mathrm{Lin}\, U \cup V$

$U = \mathrm{Lin}\{u_1, u_2\}$

$V = \mathrm{Lin}\{v_1, v_2, v_3\}$

$U+V = \mathrm{Lin}\{\underline{u_1, u_2, v_1, v_2, v_3}\}$

$$
\begin{array}{ccccc} 1 & 2 & 3 & & \\ \end{array}
$$
$$\left[u_1^T \; u_2^T \; v_1^T \; v_2^T \; v_3^T \right] \rightsquigarrow \text{Gaussove red. oblike}$$

$$
\begin{bmatrix} 1 & 1 & -1 & 3 & 2 \\ 1 & 0 & -1 & 2 & 1 \\ 1 & 0 & 0 & 2 & 2 \\ 1 & 0 & 0 & 2 & 2 \\ 1 & -1 & 1 & 1 & 2 \end{bmatrix}
\xrightarrow{-1}
\begin{bmatrix} 1 & 0 & 0 & 2 & 2 \\ 1 & 0 & -1 & 2 & 1 \\ 1 & 1 & -1 & 3 & 2 \\ 1 & -1 & 1 & 1 & 2 \end{bmatrix}
\rightarrow
$$

$$
\begin{bmatrix} 1 & 0 & 0 & 2 & 2 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 \end{bmatrix}
\rightarrow
\begin{bmatrix} 1 & 0 & 0 & 2 & 2 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\rightarrow
\begin{array}{c}
\begin{bmatrix} 1 & 0 & 0 & 2 & 2 \\ 0 & 1 & 0 & 1 & 1 \\ & & 1 & 0 & 1 \end{bmatrix} \\
\begin{array}{ccccc} 1 & 2 & 3 & & \end{array}
\end{array}
$$

$\{u_1, u_2, v_1\}$ so baza vsote

$$\alpha_1 u_1 + \alpha_2 u_2 + \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 = 0$$

Med rešitvami enačbe obstaja zveza $\rightsquigarrow$

$$\alpha_1 + 2\beta_2 + 2\beta_3 = 0$$
$$\alpha_2 + \beta_2 + \beta_3 = 0$$
$$\beta_1 + \beta_3 = 0$$

$$-(2\beta_2 + 2\beta_3)u_1 - (\underset{0}{\beta_2 + \beta_3})u_2 - \underset{0}{\beta_3}\, v_1 + \beta_2 v_2 + \beta_3 v_3 = 0$$

$$\beta_2 = 0, \quad \beta_3 = 0$$

1) u_1, u_2, v_1 so lin neodvisni

$\alpha_1 u_1 + \alpha_2 u_2 + \beta_1 u_3 = 0$ Dobimo iz zgornje enačbe za $\beta_2 = \beta_3 = 0$

Potem so tudi $\alpha_1 = \alpha_2 = \beta_1 = 0$

3) $v_2, v_3 \in \mathrm{Lin}\{u_1, u_2, v_3\}$

V zgornjo enačbo vstavimo $\beta_2 = 1, \beta_3 = 0$ in $\beta_2 = 0, \beta_3 = 1$.

$$v_2 = 2u_1 + u_2$$
$$v_3 = 2u_1 + u_2 + v_1$$

$$V = \mathrm{Lin}\{x_1 \dots x_n\} \qquad U = \mathrm{Lin}\{y_1, \dots, y_m\}$$

$v \in V \cap U$, ko ga lahko izrazimo kot

$$v = \alpha_1 x_1 + \dots \alpha_n x_n \quad \text{in} \quad v = -(\beta_1 y_1 + \dots + \beta_m y_m)$$

$$0 = v - v = \alpha_1 x_1 + \dots + \alpha_n x_n + \beta_1 y_1 \dots \beta_m y_m$$

$$\begin{bmatrix} x_1 & \cdots & x_n & y_1 & \cdots & y_m \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \\ \beta_1 \\ \vdots \\ \beta_m \end{bmatrix}$$

Iščemo za katere $[\alpha_1 \dots \alpha_n]$ obstajajo $[\beta_1, \dots, \beta_m]$, da bo

$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \\ \beta_1 \\ \vdots \\ \beta_m \end{bmatrix}$ rešitev enačbe

Enačbo smo rešili že prej

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 2 \\ 0 & 1 & 0 & 1 & 1 \\ & & 1 & 0 & 1 \end{bmatrix}$$

$$\alpha_1 + 2\beta_2 + 2\beta_3 = 0$$
$$\alpha_2 + \beta_2 + \beta_3 = 0$$
$$\beta_1 + \beta_3 = 0$$

$$\alpha_1 = -2(\beta_2 + \beta_3)$$
$$\alpha_2 = -(\beta_2 + \beta_3)$$
$$\beta_1 = -\beta_3$$

$$\alpha_1 = 2\alpha_2$$

$2u_1+u_2 \in \text{Lin}\{v_1, v_2, v_3\}$

$\alpha_1 u_1 + \alpha_2 u_2 \notin$ če $\alpha_1 \neq 2\alpha_2$

$$\underbrace{\dim(U+V)}_{3} = \underbrace{\dim(U)}_{2} + \underbrace{\dim(V)}_{2} - \underbrace{\dim(U\cap V)}_{1}$$

$v_3 = v_1 + v_2$

$U \cap V = \text{Lin}\{2u_1+u_2\}$

6. Za vektorska podprostora U, V vektorskega prostora W dokaži ekvivalence

   (i) $U + V = V \iff U \subseteq V$

   (ii) $U \cap V = V \iff V \subseteq U$

   (iii) $U + V = U \cup V \iff (U \subseteq V$ ali $V \subseteq U)$.

   Pokaži, da je $U \cup V$ vektorski podprostor natanko tedaj, ko je $(U \subseteq V$ ali $V \subseteq U)$.

$\Rightarrow$

i) Privzamemo $U+V=V$

⊛ $\forall u \in U, v \in V \quad u+v \in V$

Želimo pokazati $\forall u \in U \Rightarrow u \in V$

$u \in U \quad 0 \in V \xRightarrow{\text{po ⊛}} u+0=u \in U$

$\Leftarrow$ Privzamemo $U \subseteq V$ želimo pokazati, da je $U+V=V$.

$U+V \supseteq V$ ✓

$U+V \subseteq V \quad u+v \in U+V \quad u \in U \subseteq V \quad \underset{\text{zaprtost za seštevanje}}{u,v\in U \Downarrow} \; u+v \in U$

ii) $U\cap V = V \iff V\subseteq U$

$\Leftarrow$ Privzamo $\underline{V\subseteq U}$ Želimo $U\cap V = V$

$U\cap V\subseteq V$ ✓

$V\subseteq U\cap V$

$V\subseteq U\cap V$

$v\in V \Rightarrow v\in U$ $\qquad$ $v\in V$ in $v\in U \overset{def}{\iff} v\in V\cap U$

$\Rightarrow$

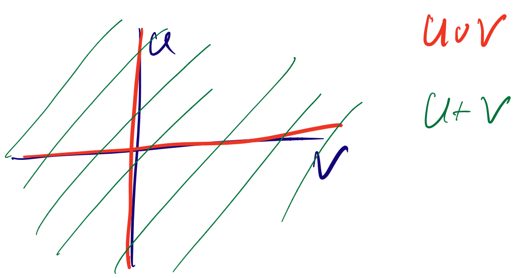
Privzamo $U\cap V=V$

$V\subseteq U$ $\qquad$ $v\in V$ želimo pokazati $v\in U$

$v\in V=U\cap V$ $\qquad$ $v\in U$ in $v\in V \Rightarrow \underline{v\in U}$

iii)

$U+V=U\cup V \iff (U\subseteq V$ ali $V\subseteq U)$



$A \Rightarrow B$ ali C

Privzamo A in $\neg B$ in pokažemo C

in Privzamemo A in $\neg C$ in pokažemo B

$\Rightarrow$

$\underline{U+V=U\cup V} \Rightarrow (\underline{U\subseteq V}$ ali $\underline{V\subseteq U})$

Privzamemo $\underline{U+V=U\cup V}$ in $\underline{U\not\subseteq V}$ želimo pokazati $\underline{V\subseteq U}$

$V\not\subseteq U$ $\qquad$ $U\subseteq V$ ☆

$v \in V$ želimo pokazati $v \in U$

Za vsak $u \in U$ je $u+v \in U+V = U \cup V$

$u+v \in U$ ali $u+v \in V$ Obstaja $u \in U$ $u \notin V$ $(u \notin V)$

$u+v \notin V$ $\left(\begin{array}{l} \text{Če je} \\ u+v \in V \Rightarrow \underbrace{u+v-v}_{u} \in V \quad \times \end{array} \right)$

$u+v \in U$ $v = u+v-u \in U \Rightarrow v \in U$ ✓

Simetrično ⊛

$\Leftarrow$ $U \subseteq V$ ali $V \subseteq U \Rightarrow U+V = U \cup V$

$U \cup V \subseteq U+V$ ✓

$\underline{U+V \subseteq U \cup V}$ — ① $U \subseteq V \overset{(i)}{\Longrightarrow} U+V = V$

$U \cup V = V$

② $V \subseteq U \overset{(i)}{\Longrightarrow} V+U = U$

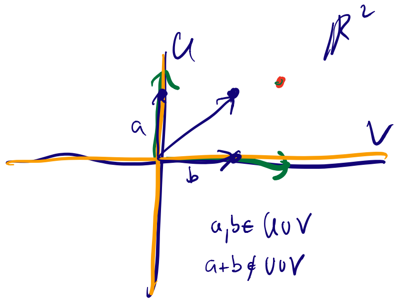
$\|$

$V \cup U = U$

$\underline{U \cup V \text{ je vektorski prostor}} \Leftrightarrow \underline{U \subseteq V \text{ ali } V \subseteq U}$

$U+V$ je najmanjši v.p., ki vsebuje U in V

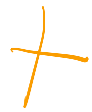
Primer, ko unija ni vektorski prostor.



$a, b \in U \cup V$

$a + b \notin U \cup V$

$\underline{U + V} = \mathbb{R}^2$

$U \cup V =$ 

$\downarrow$

ni zaprto za seštevanje