- $a_n > 0$
- $\lim_{n \to \infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) =: r$

a) Če je $r > 0 \Rightarrow \sum_{n \geq 0} a_n < \infty$

b) Če je $r < 0 \Rightarrow \lim_{n \to \infty} a_n = \infty$

a) Raabejev izrek:

Če obstaja $\lim_{n \to \infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) =: r$, potem velja:

- če je $r > 1$, vrsta konvergira
- Če je $r < 1$, vrsta divergira

$\boxed{r > 1}$, skripta, str. 235-236

$\boxed{r > 1}$

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) > r' \quad ; 1 < r' < r$$

za vse dovolj velike n

$$\frac{a_n}{a_{n+1}} - 1 > \frac{r'}{n} \quad *$$

Vzamemo s; $1 < s < r$

$$\lim_{n \to \infty} \frac{\left(1 + \frac{1}{n}\right)^s - 1}{\frac{1}{n}} = s < r$$

n

$\hookrightarrow \lim_{x\to 0} \frac{(1+x)^s - 1}{x} \ldots$

$\frac{a_n}{a_{n+1}} > \left(1+\frac{1}{n}\right)^s = \frac{\left(\frac{1}{n}\right)^s}{\left(\frac{1}{n+1}\right)^s}$

$\left(1+\frac{1}{n}\right)^s - 1 < \frac{r}{n}$; za dovolj velike n

$\frac{a_{n+1}}{a_n} < \frac{\left(\frac{1}{n+1}\right)^s}{\left(\frac{1}{n}\right)^s}$

S pomočjo tega:

$\frac{a_n}{a_{m-1}} = \frac{a_m}{a_{m-1}} \frac{a_{m+1}}{a_m} \frac{a_{m+2}}{a_{m+1}} \cdots \frac{a_n}{a_{n-1}}$

$< \frac{\left(\frac{1}{n}\right)^s}{\left(\frac{1}{m-1}\right)^s}$

oz. $a_n < \frac{a_{m-1}}{\left(\frac{1}{m-1}\right)^s} \cdot \left(\frac{1}{n}\right)^s$

$\Rightarrow \sum a_n \leq \sum \left(\frac{1}{n}\right)^s$: ta vrsta konvergira

$r < 0$

$$r = \lim_{n \to \infty} \left(\frac{a_n}{a_{n+1}} - 1 \right) \cdot n$$

Sledi: za vse dovolj velike n je

$\frac{a_n}{a_{n+1}} - 1 < 0$, $a_n < a_{n+1}$ za vse

dovolj velike n

$$\Rightarrow \lim_{n \to \infty} a_n \neq 0$$

$\lim_{n \to \infty} a_n = \infty$ dobimo iz tega,

$$\frac{a_n}{a_{n+1}} - 1 \approx \frac{1}{n} \quad \text{!!}$$

Integrali s trigonometričnimi funkcijami

$I = \int \cos^m x \sin^n x \, dx$; m ali n liha

- m liha

$m - 1 = 2k$

$\cos^2 x = 1 - \sin^2 x$

$t = \sin x$

$dt = \cos x \, dx$

$I = \int \cos x \cdot \cos^{2k} x \sin^n x \, dx =$

$= \int (1 - t^2) \cdot t^n \, dt = \ldots$

$\int \cos^m x \sin^n x \, dx$; m in n soda

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x =$

$= 1 - 2\sin^2 x =$

$= \cos^2 x - 1$

$$\int \sin^2 x \, dx = \frac{1}{2}\int 2\sin^2 x \, dx =$$

$$= \frac{1}{2}\int 2\sin^2 x - 1 + 1 \, dx =$$

$$= \frac{1}{2}\int dx - \frac{1}{2}\int 1 - 2\sin^2 x \, dx =$$

$$= \frac{1}{2}x - \frac{1}{2}\int \cos 2x \, dx =$$

$$= \underline{\frac{1}{2}x - \frac{1}{4}\sin 2x + C}$$

$t = 2x$

$dt = 2\,dx$

$$\int \frac{dx}{1 + \sin^2 x} \, dx$$

$$\int R\left(\cos^2 x, \sin^2 x, \sin x \cos x\right) dx$$

↳ rac. funkcija

V tem primeru:

$t = tg\, x$

$dt = \frac{1}{\cos^2 x} dx = \left(t^2 + 1\right) dx$

$dx = \frac{dt}{1+t^2}$

$$\underbrace{\cos^2 x} = \frac{1}{\frac{1}{\cos^2 x}} = \frac{1}{tg^2 x + 1} = \underbrace{\frac{1}{t^2+1}}$$

$$tg^2 x + 1 = \frac{1}{\cos^2 x}$$

$$\underbrace{\sin^2 x} = 1 - \cos^2 x = 1 - \frac{1}{t^2+1} =$$

$$= \frac{t^2+1-1}{t^2+1} = \underbrace{\frac{t^2}{t^2+1}}$$

$$\underbrace{\sin x \cos x} = \frac{\overbrace{\sin x}^{tg\, x} \cos^2 x}{\cos x} = \underbrace{\frac{t}{t^2+1}}$$

$$\int \frac{dx}{1+\sin^2 x} = \int \underbrace{\frac{dt}{1+t^2}}_{dx} \cdot \underbrace{\frac{1}{1+\frac{t^2}{t^2+1}}} =$$

$$= \int \frac{t^2+1}{\underbrace{t^2+1+t^2}} \cdot \frac{1}{t^2+1}\, dt =$$

$$= \int \frac{dt}{2t^2+1} = \frac{1}{\sqrt{2}} \int \frac{ds}{s^2+1} =$$

$$\downarrow$$

$$s = \sqrt{2}\,t$$

$$ds = \sqrt{2}\,dt$$

$$= \frac{1}{\sqrt{2}} \operatorname{arctg}(\sqrt{2}\,t) + C =$$

$$= \frac{1}{\sqrt{2}} \operatorname{arctg}(\sqrt{2}\operatorname{tg} x) + C$$

$$\int R(\sin x, \cos x)\,dx \rightsquigarrow \int R\left(\sin^2\frac{x}{2}, \cos^2\frac{x}{2}, \sin\frac{x}{2}\cos\frac{x}{2}\right)$$

$$t = \tan\frac{x}{2}$$

$$\sin x = \frac{2t}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$dx = \frac{2dt}{1+t^2}$$

$$\hookrightarrow x = 2\arctan t$$

$$\int \frac{dx}{5+\sin x} \underset{t = tg\frac{x}{2}}{=} \int \frac{2\,dt}{1+t^2}\,\frac{1}{5+\frac{2t}{1+t^2}} =$$

$$= 2\int \frac{dt}{1+t^2}\,\frac{1+t^2}{5(1+t^2)+2t} =$$

$$= 2\int \frac{dt}{5t^2+2t+5} =$$

$$= \frac{2}{5}\int \frac{dt}{t^2+\frac{2}{5}t+1} = \circledast$$

$$\underbrace{t^2+\frac{2}{5}t}+1 \simeq s^2 + a$$

$$t^2+\frac{2}{5}t+\underbrace{\frac{1}{25}} = \left(t+\frac{1}{5}\right)^2$$

$$\underbrace{t^2+\frac{2}{5}t+1}+\underbrace{\frac{1}{25}}-\frac{1}{25} =$$

$$= \left(t+\frac{1}{5}\right)^2+\frac{24}{25}$$

$$\circledast = \frac{2}{5}\int \frac{dt}{\left(t+\frac{1}{5}\right)^2+\frac{24}{25}} =$$

$$= \frac{2}{5} \cdot \frac{25}{24} \int \frac{dt}{\frac{25}{24}\left(t+\frac{1}{5}\right)^2 + 1} =$$

$$= \frac{5}{12} \int \frac{dt}{\left(\frac{5}{\sqrt{24}} t + \frac{1}{\sqrt{24}}\right)^2 + 1} =$$

$$\sqrt{24} = 2\sqrt{6}$$

$$s = \frac{5}{2\sqrt{6}} t + \frac{1}{2\sqrt{6}}$$

$$ds = \frac{5}{2\sqrt{6}}\, dt$$

$$= \frac{5}{12} \cdot \frac{2\sqrt{6}}{5} \int \frac{ds}{s^2+1} = \ldots$$

$$= \frac{\sqrt{6}}{6} \operatorname{arctg}(s) + C$$

$$s \rightarrow t \rightarrow x$$

$$\int \sin 6x \cos 13x \, dx = ? = \cancel{t}$$

$$\sin 6x \cos 13x = \frac{1}{2}\left[\sin 19x + \sin(-7x)\right]$$

ADICIJSKI izreki:

$$*\quad \left.\begin{array}{l} \sin(x+y) = \sin x\cos y + \cos x\sin y \\ \sin(x-y) = \sin x\cos y - \cos x\sin y \end{array}\right]$$

$$\sin(x+y) - \sin(x-y) = 2\cos x\sin y \quad / \cdot \tfrac{1}{2}$$

$$\sin(x+y) + \sin(x-y) = 2\sin x\cos y$$

$$* = \frac{1}{2}\int \sin 19x - \sin 7x \, dx =$$

$$= \frac{1}{2}\left(-\frac{\cos(19x)}{19} + \frac{\cos(7x)}{7}\right) + C$$

<u>Integrali tipa</u> $\displaystyle\int \frac{Mx+N}{x^2+px+q}\,dx$

$M, N, p, q \in \mathbb{R}$; $\underbrace{p^2-4q}_{D:=} < 0$

tj. imenovalec je nerazcepen

$$x^2+px+q = (\underbrace{x+\tfrac{p}{2}}_{t})^2 \underbrace{- \frac{p^2}{4} + q}_{\frac{-D}{4}}$$

$$=$$

$$=: t^2 + h^2 \qquad h = \sqrt{\frac{4q - p^2}{4}}$$

$$\int \frac{Mx + N}{x^2 + px + q}\, dx = \int \frac{Mt - \frac{p}{2}M + N}{t^2 + h^2}\, dt =$$

$$t = x + \frac{p}{2}$$

$$dx = dt$$

$$x = t - \frac{p}{2}$$

$$= M \int \frac{t}{t^2 + h^2}\, dt + \left(N - \frac{p}{2}M\right) \int \frac{dt}{t^2 + h^2}$$

$$\int \frac{t}{t^2 + h^2}\, dt = \frac{1}{2} \ln s + C$$

$$s := t^2 + h^2$$

$$ds = 2t\, dt$$

$$\int \frac{dt}{t^2 + h^2} = \frac{1}{h^2} \int \frac{dt}{\left(\frac{t}{h}\right)^2 + 1} = \left(N - \frac{p}{2}M\right) \frac{1}{h} \operatorname{arctg}\left(\frac{t}{h}\right) + C$$

$$k = \sqrt{\frac{4q-p^2}{4}} \qquad s = \frac{t}{k}$$

$$t = x + \frac{p}{2} \qquad ds = \frac{dt}{k}$$

$$dt = k\,ds$$

$$\int$$

$$\Rightarrow$$

$$\boxed{\int \frac{Mx+N}{x^2+px+q}\,dx = \frac{M}{2}\ln\left(\overbrace{x^2+px+q}^{t^2+k^2}\right) + \frac{2N-pM}{\sqrt{4q-p^2}}\,\mathrm{arg}\left(\frac{2x+p}{\sqrt{4q-p^2}}\right) + C}$$

$$\frac{1}{k} = \frac{2}{\sqrt{4q-p^2}}$$

$$\int \frac{2x+1}{(x-1)(x^2+x+1)}\,dx =$$

$$\frac{2x+1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} =$$

$$= \frac{A(x^2+x+1) + Bx(x-1) + C(x-1)}{(x-1)(x^2+x+1)}$$

Primerjava koeficientov

$x^2: \quad 0 = A + B \quad \Rightarrow \quad B = -A$

$x: \quad 2 = A - B + C \quad \Rightarrow \quad 2 = 2A + C$

$1: \quad 1 = A - C$

$2 = 2A + A - 1$

$\Rightarrow C = A - 1 \qquad \Rightarrow \underline{A = 1}$

$\Rightarrow B = -1, \quad C = 0$

$$* = \int \frac{dx}{x-1} - \int \frac{x}{x^2+x+1}\,dx = *$$

$$\int \frac{x}{x^2+x+1}\,dx = X$$

$t = x^2 + x + 1$

$dt = x + 1 \; dx$

$$\int \frac{x}{x^2+x+1} = \frac{1}{2}\ln\left(x^2+x+1\right) + \frac{-1}{\sqrt{3}}\,\mathrm{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

$$Mx + N = x \Rightarrow M = 1,\ N = 0$$

$$x^2 + px + q = x^2 + x + 1 \Rightarrow p = q = 1$$

$$\boxed{\int \frac{Mx+N}{x^2+px+q}\,dx = \frac{M}{2}\ln\left(x^2+px+q\right) + \frac{2N-pM}{\sqrt{4q-p^2}}\,\mathrm{arg}\left(\frac{2x+p}{\sqrt{4q-p^2}}\right) + C}$$

1 2

$$\cancel{Ix}\ \cancel{I} = \ln|x-1| - \frac{1}{2}\ln\left|x^2+x+1\right| + \frac{1}{\sqrt{3}}\,\mathrm{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

Eulerjeva substitucija

$$\int R\left(x, \sqrt{ax^2+bx+c}\right); \quad a \neq 0$$

a) $a > 0$

b) $ax^2 + bx + c = a(x-\alpha)(x-\beta)$; $\alpha \neq \beta$

c) $c \geq 0$

a) $a > 0$

Uvede se novo spremenljivko t, da velja:

$$\sqrt{ax^2 + bx + c} = x\sqrt{a} + t$$

$$\Rightarrow \quad x = \frac{t^2 - c}{b - 2t\sqrt{a}}$$

$$dx = \ldots$$

$$I = \int R(t)\, dt$$

$$\int \frac{dx}{\sqrt{x^2+x+1}} = \int \frac{\cancel{(1-2t)}\; \cancel{2(t-t^2-1)}}{\cancel{(t-t^2-1)}\;(1-2t)^{\cancel{2}}}\, dt =$$

$$x + t = \sqrt{x^2+x+1} \quad \rightsquigarrow \quad t = \sqrt{x^2+x+1} - x$$

$$x = \frac{t^2 - 1}{1 - 2t}$$

$$dx = \frac{2t(1-2t) - (-2)(t^2-1)}{(1-2t)^2}\, dt$$

$$= \frac{-2t^2 + 2t - 2}{(1-2t)^2}\, dt =$$

$$= 2 \, \frac{-t^2 + t - 1}{(1-2t)^2}\, dt$$

$$\sqrt{x^2+x+1} = x + t = \frac{t^2-1}{1-2t} + t =$$

$$= \frac{t - t^2 - 1}{1-2t}$$

$$= 2\int \frac{dt}{1-2t} = \frac{2}{-2} \ln|1-2t| + C$$

$$= -1 \ln\left|1 - 2\cdot\left(\sqrt{x^2+x+1} - x\right)\right| + C$$

$$t = \sqrt{x^2+x+1} - x$$

b) $\underline{ax^2 + bx + c = a(x-\alpha)(x-\beta)}$; $\alpha \neq \beta$

Nova spremenljivka t:

$$\boxed{\sqrt{ax^2+bx+c} = (x-a)t}$$

$$\boxed{x = \frac{\alpha t^2 - a\beta}{t^2 - a}}$$

dx ...

rac. funkcije

Spet dobimo rac. funkcijo $\int Q(t)\,dt$

$$\int \frac{dx}{(x-2)\sqrt{3x-x^2-2}} =$$

$3x - x^2 - 2 =$

$= -(x^2-3x+2)$

$$\sqrt{3x - x^2 - 2} = t(x-1) \qquad = -\underset{\beta}{(x-2)}\underset{\alpha}{(x-1)}$$

$$x = \frac{t^2+2}{t^2+1}$$

$$dx = \underbrace{\dots}\, dt$$

$$\int \frac{dx}{\left(\frac{t^2+2}{t^2+1} - 2\right) t \left(\frac{t^2+2}{t^2+1} - 1\right)} = \dots$$

- $\boxed{c \geq 0}$

$$\boxed{\sqrt{ax^2+bx+c} = xt + \sqrt{c}}$$

...

$$\boxed{x = \frac{b - 2t\sqrt{c}}{t^2 - a}}$$

D. N.

$f_0 = 0, \quad f_1 = f_2 = 1$

$f_n = f_{n-1} + f_{n-2}$ za $n \geq 2$

a) $f_n \perp f_{n+1}$

$n = 1$

$f_1 = f_2 = 1 \Rightarrow f_1 \perp f_2$ ✓

$n \Rightarrow n+1$

$f_n \perp f_{n+1}$] I.P.

$f_{n+1} \perp f_{n+2}$

Celo število m deli f_{n+1} in f_{n+2}, sledi:

$$f_{n+2} = f_{n+1} + f_n \Rightarrow f_n = \underbrace{f_{n+2} - f_{n+1}}_{= m\, k}$$

$$\Rightarrow m \mid f_n, \quad m \mid f_{n+1}$$

$\Rightarrow \quad m = 1$

c) $\forall m, n$:

$$\boxed{f_{m+n} = f_{m-1} f_n + f_m f_{n+1}}$$

$\Rightarrow \quad f_{2n} = f_n f_{n-1} + f_n f_{n+1}$

indukcija po n

$\underline{n=1}$ $\qquad \frown\!\!) \quad f_1 = f_2 = 1$

$$f_{m+1} = f_{m-1} + f_m =$$

$$= f_{m-1} f_1 + f_m f_2$$

$\underline{n=2}$

$$f_{m+2} = f_{m+1} + f_m = f_{m-1} + f_{m-1} + f_m$$

$$= 2 f_m + 1 \cdot f_{m-1}$$

$$\underset{f_{2+1}}{\overset{\shortparallel}{2}} \qquad \underset{f_2}{\overset{\shortparallel}{1}}$$

Splošen n !

IP : Trditev velja za n in $n-1$

Dokazujemo : Trditev velja za $n+1$

IP :

$$\left.\begin{aligned} f_{m+n} &= f_{m-1} f_n + f_m f_{n-1} \\ f_{m+n-1} &= f_{m-1} f_{n-1} + f_m f_n \end{aligned}\right\} \quad \begin{aligned} &(\text{i.p. za } n) \\ &(\text{i.p. za } n-1) \end{aligned}$$

$$f_{m+n+1} = f_{m+n} + f_{m+n-1} \overset{IP}{=}$$

$$= f_{m-1} f_n + f_m f_{n-1} + f_{m-1} f_{n-1}$$

$$+ f_m f_n =$$

$$= \underbrace{f_{m-1} f_{n+1}}_{} + f_m \underbrace{f_{n+2}}_{}$$

$$f_n + f_{n-1} \qquad\qquad f_{n+1} + f_n$$

3) $(a_n)_n$ zaporedje realnih št.;

$|a_{n+1} - a_n| < c^n$; $0 < c < 1$,

za vsak n

Dokaz: zaporedje je konvergentno.

Zaporedje je Cauchyjevo

$\forall \varepsilon > 0 : \exists N \ \forall m_1, m_2 > N$:

$|a_{m_1} - a_{m_2}| < \varepsilon$ $\quad m_2 > m_1$:

$|a_{m_2} - a_{m_1}| =$

$= |a_{m_2} - a_{m_2-1} + a_{m_2-1}$

$- a_{m_2-2} + a_{m_2-2}$

$- \dots$

$- a_{m_1+1} + a_{m_1+1} - a_{m_1}|$

$\leq |a_{m_2} - a_{m_2-1}| + |a_{m_2-1} - a_{m_2-2}| +$

$$\ldots + |a_{m_1+1} - a_{m_1}|$$

$$< c^{m_2-1} + c^{m_2-2} + \ldots + c^{m_1}$$

$$= \sum_{n=m_1}^{m_2-1} c^n < \sum_{n=N}^{\infty} c^n < \varepsilon$$

$0 < c < 1 \Rightarrow \sum_{n \geq 0} c^n$ konvergira

Torej $\exists N$, da je

$$\sum_{n=N}^{\infty} c^n < \varepsilon$$

Torej je zaporedje Cauchyjevo, torej konvergira.

$\lim_{n\to\infty} |a_{n+1} - a_n| = 0 \Rightarrow \{a_n\}_n$ konvergira?

NE:

$$a_n := \sum_{k=1}^{n} \frac{1}{k}$$

$$|a_{n+1} - a_n| = \frac{1}{n+1} \xrightarrow{n \to \infty} 0$$

Vseeno: $\{a_n\}_n$ divergira

4) $a_{n+1} = \sqrt{3a_n}$, $a_0 = 2$

Dokaži, da konvergira in izračunaj limito.

1. način: rekurzivno

- izr. kand. za limito: $0, 3$
- $a_n < 3$ za vsak n (ind. po n)
- $a_{n+1} \geq a_n$

$$\sqrt{3a_n} \geq a_n \quad /()^2$$

$$3a_n \geq a_n^2$$

$$a_n(3 - a_n) \geq 0$$

$a_n > 0$

$3 > a_n$

2. ~~Eksplicitno~~ Eksplicitno

Za $n \geq 1$:

$$a_n = 3^{\left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{n-1} + \dots + \left(\frac{1}{2}\right)} \cdot 2^{\left(\frac{1}{2}\right)^n}$$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} 3^{\underbrace{\left(\frac{1}{2}\right)^n + \dots + \left(\frac{1}{2}\right)}_{\xrightarrow{n \to \infty} 1}} \cdot 2^{\underbrace{\left(\frac{1}{2}\right)^n}_{\to 1}} = 3$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right) = \frac{\frac{1}{2}}{1 - \frac{1}{2}}$$

$$= 1$$

5) Za katera $a \in \mathbb{R}$ konvergira

$$\sum_{n=2}^{\infty} \left(\frac{a^2+9}{6a}\right)^n \frac{1}{\ln(n^2)}$$

$$\left|\frac{a^2+9}{6a}\right| > 1 \Rightarrow \text{vrsta divergira}$$

$$\lim_{n \to \infty} \left(\frac{a^2+9}{6a}\right)^n \cdot \frac{1}{2\ln n}$$ ... limita ne obstaja,

ker gredo abs. vrednosti proti ∞

$\left|\frac{a^2+9}{6a}\right| < 1 \Rightarrow$ vrsta abs. konvergira

$$\sum_{n=2}^{\infty} \left|\frac{a^2+9}{6a}\right|^n \frac{1}{\ln(n^2)} \leq \sum_{n=2}^{\infty} \left|\frac{a^2+9}{6a}\right|^n$$

$$< \infty$$

Po prim. kriteriju konvergira tudi naša vrsta

$$\left|\frac{a^2+9}{6a}\right| = 1$$

$\sum_{n=2}^{\infty} \frac{1}{\ln(n^2)}$ ... divergira

$\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln(n^2)}$ ... pogojno konvergira

a^2+9

$\overline{\quad 6a \quad} = 1$ ... vrsta divergira

$= -1$ ... vrsta konvergira

rešite kv. enačbo / neenačbe, da dobite konkretne a, za katere vrsta konvergira

$$\left| \frac{a^2+9}{6a} \right| > 1$$

$\Leftrightarrow$

$$|a^2+9| > 6|a|$$

$$\overset{\shortparallel}{a^2+9}$$

$\hookrightarrow$ $a>0$, $a<0$

Dobite:

$\Rightarrow$ vrsta div. za
$a>0$, $a \neq 3$
$a<0$, $a \neq -3$

Vrsta pog. konvergira za $a = -3$

$$\frac{a^2+9}{6a} = -1$$

Vrsta divergira za $a = 3$

$$\lim_{x \to \infty} (1+x)^{\frac{1}{x}} = \lim_{x \to \infty} e^{\underbrace{\frac{1}{x} \ln(1+x^2)}}$$