Naj bo f integrabilna na intervalu $[-a, a]$ za nek $a > 0$. Pokaži:

- $\int_{-a}^{a} f(t)\,dt = 0$; f liha
- $\int_{-a}^{a} f(t)\,dt = 2\int_{0}^{a} f(t)\,dt$; f soda

$$\int_{-a}^{a} f(t)\,dt = \int_{-a}^{0} f(t)\,dt + \int_{0}^{a} f(t)\,dt =$$

$u = -t$

$du = -dt$

$$= -\int_{a}^{0} f(-u)\,du + \int_{0}^{a} f(t)\,dt =$$

$$= \int_{0}^{a} f(-u)\,du + \int_{0}^{a} f(t)\,dt = *$$

f liha: $f(-u) = -f(u)$

$$* = \int_{0}^{a} -f(u)\,du + \int_{0}^{a} f(t)\,dt =$$

$$= \underline{\underline{0}}$$

f soda: $\quad f(-u) = f(u)$

$$\ast = \int_0^a f(u)\,du + \int_0^a f(t)\,dt \checkmark$$

$0 < a < b$

$f : [a, b] \longrightarrow (0, \infty)$ zvezna

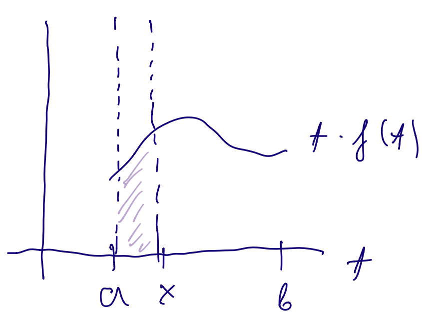
$$F(x) = \int_a^x f(t)\,dt \; ; \quad G(x) = \int_a^x t\,f(t)\,dt$$

$\frac{F}{G}(x)$ definiran na (a, b) in $\frac{F}{G}$ je strogo padajoča na (a, b).

F, G sta oba def. na (a, b)

• zvezna funkcija na zaprtem intervalu je integrabilna.

za vsak $x \in (a, b)$: $\quad G(x) = \int_a^x \underbrace{t\,f(t)}_{\substack{\vee \\ 0}}\,dt \neq 0$



za $t \in (a, b)$?

$\Box$: $0 < a < b$

$\Rightarrow t > 0$, $f(t) > 0$

$$G(x) > 0 \quad \text{za} \quad x \in (a,b)$$

$\frac{F}{G}(x)$ je vedno definirana na (a,b)

$$F(x) = \int_a^x f(t)\,dt$$

$$\left(\frac{F}{G}\right)'(x) = \frac{F'G - G'F}{G^2} =$$

$$F'(x) = f(x)$$

$$= \frac{f(x)\int_a^x t f(t)\,dt - x f(x)\int_a^x f(t)\,dt}{G^2} =$$

$$= \frac{f(x)\left[\int_a^x t f(t)\,dt - \int_a^x x f(t)\,dt\right]}{G^2} =$$

$$= \frac{\underbrace{f(x)}_{>0}\int_a^x \underbrace{f(t)}_{>0}\underbrace{(t-x)}_{<0}\,dx}{\underbrace{G^2}_{>0}} < 0 \qquad t \in (a,x)$$

NEDOLOČENI INTEGRALI

$$\int f(x)\,dx = \{g \,;\, g' = f\}$$

$$= g(x) + C$$

Substitucija (uvedba nove spremenljivke)

$$\int f(g(x))\, g'(x)\, dx = \int f(t)\, dt$$

$t := g(x)$

$dt = g'(x)\, dx$

$$\int \frac{x^2}{(x-1)^{100}}\, dx = \int \frac{(t+1)^2}{t^{100}}\, dt =$$

$t := x - 1$

$dt = dx$

$$= \int \frac{t^2 + 2t + 1}{t^{100}}\, dt =$$

$a \neq -1$

$\int x^a dx = \frac{x^{a+1}}{a+1}$

$$= \int \frac{1}{t^{98}}\, dt + 2\int \frac{1}{t^{99}}\, dt + \int \frac{1}{t^{100}}\, dt$$

$\frac{1}{t^{98}} = t^{-98}$, $\frac{1}{t^{99}} \to t^{-99}$, $\frac{1}{t^{100}} \to t^{-100}$

$$= \frac{t^{-97}}{-97} + 2\,\frac{t^{-98}}{-98} + \frac{t^{-99}}{-99} + C =$$

$$= \frac{(x-1)^{-97}}{-97} + 2\,\frac{(x-1)^{-98}}{-98} + \frac{(x-1)^{-99}}{-99} + C$$

$$\int x\, e^{x^2}\, dx = \frac{1}{2}\int e^t\, dt = \frac{e^{x^2}}{2} + C$$

$$t = f(x) \qquad t = x^2$$

$$dt = \underbrace{f'(x)}\, dx \qquad dt = 2\cancel{x}\, dx$$

$$\int \frac{dx}{\sqrt{x}\,(1+\sqrt[3]{x})} = \int \frac{6t^5\, dt}{t^3(1+t^2)} = 6\int \frac{t^2}{1+t^2}\, dt = \cancel{*}$$

$$t = \sqrt[6]{x} = x^{\frac{1}{6}} \quad \rightarrow \quad x = t^6$$

$$dt = \frac{1}{6}x^{-\frac{5}{6}}\, dx \qquad dx = 6t^5\, dt$$

$$\frac{t^2}{1+t^2} = \frac{t^2+1-1}{1+t^2} =$$

$$\frac{t^2}{1+t^2} = \frac{1}{1+t^2} + \underline{\qquad} \qquad = 1 - \frac{1}{1+t^2}$$

$$\frac{1}{1+\frac{1}{t^2}}$$

$$* = 6\int 1 - \frac{1}{1+t^2}\, dt = 6\int dt - 6\int \frac{dt}{1+t^2} =$$

$$\frac{t^2}{1+t^2}$$

$$= 6t - 6\arctan(t) + C =$$

$$= \underline{\underline{6x^{\frac{1}{6}} - 6\arctan\left(x^{\frac{1}{6}}\right) + C}}$$

$$\int \frac{dx}{\left(\sqrt[3]{x} - \sqrt[4]{x}\right)^2} = \int \frac{12t^{11}}{(\underbrace{t^4 - t^3}_{\text{izp. } t^3})^2}\,dt =$$

$$t = \sqrt[12]{x}$$

$$x = t^{12}$$

$$dx = 12t^{11}\,dt$$

$$= 12\int \frac{t^{11}}{t^6(t-1)^2}\,dt = 12\int \frac{t^5}{(t-1)^2}\,dt =$$

$$u = t-1$$

$$du = dt$$

$$= 12\int \frac{(u+1)^5}{u^2}\,du =$$

$$= 12 \int \frac{u^5 + 5u^4 + 10u^3 + 10u^2 + 5u + 1}{u^2}\, du =$$

1 4 6 4 1

1 5 10 10 5 1

$$= 12 \left(\int u^3\, du + 5 \int u^2\, du + 10 \int u\, du + 10 \int du + 5 \int \frac{du}{u} + \int \frac{du}{u^2} \right) = \ldots$$

$$5 \int \frac{du}{u} = 5 \ln u$$

$$u = t - 1 = \sqrt[12]{x} - 1$$

Integracija po delih/per partes

$$[f(x)g(x)]' = f'(x)g(x) + g'(x)f(x) \quad / \int dx$$

$$f(x)g(x) = \int f'(x)g(x)\,dx + \int g'(x)f(x)\,dx$$

$$\int f(x)g'(x)\,dx = ?$$

$$\int f'(x)\, g(x)\, dx \quad \text{lažje}$$

$$\int f'(x)\, g(x)\, dx = f(x)\, g(x) - \int g'(x)\, f(x)\, dx$$

$$\int \underset{u}{x}\, \underbrace{e^{x}\, dx}_{dv} = \underset{u}{x}\, \underset{v}{e^{x}} - \int \underset{v}{e^{x}}\, \underset{du}{dx} =$$

$$u = x \qquad dv = e^{x}\, dx$$

$$du = dx \qquad v = e^{x}$$

$$= \underline{\underline{x e^{x} - e^{x} + C}}$$

$$\overbrace{\int e^{x} \cos x\, dx}^{I} = e^{x} \cos x + \int e^{x} \sin x\, dx = \cancel{A}$$

$$u = \cos x \qquad dv = e^{x}\, dx$$

$$du = -\sin x\, dx \qquad v = e^{x}$$

$$\int e^{x} \sin x\, dx = e^{x} \sin x - \int e^{x} \cos x\, dx$$

$$u = \sin x \qquad dv = e^{x}\, dx$$

$$du = \cos x\, dx \qquad v = e^{x}$$

$$\cancel{I} = e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$$

$$I = e^x \cos x + e^x \sin x - I$$

$$I = \frac{1}{2} e^x (\cos x + \sin x) + C$$

$$\int x^a \ln x \, dx =$$

$$u = \ln x \qquad dv = x^a$$

$$du = \frac{1}{x} dx \qquad v = \frac{x^{a+1}}{a+1} \qquad a \neq -1$$

$$= \frac{\ln x \cdot x^{a+1}}{a+1} - \int \frac{x^a}{a+1} dx =$$

$$= \frac{\ln x \cdot x^{a+1}}{a+1} - \frac{x^{a+1}}{(a+1)^2} + C \qquad (a \neq 1)$$

$$a = -1$$

$$\int \frac{\ln x}{x} dx = \int t \, dt = \frac{t^2}{2} + C = \frac{(\ln x)^2}{2} + C$$

$\hookrightarrow t = \ln x$

$dt = \frac{1}{x}\,dx$

$$\int x^5 \sqrt{x^3+1}\,dx = \cancel{*}$$

$\hookrightarrow x^5 = x^3 \cdot x^2$

$u = x^3$

$du = 3x^2\,dx$

$dv = x^2\sqrt{x^3+1}\,dx$

$v = \frac{2}{9}(x^3+1)^{3/2}$

$$\int \sqrt{x^3+1}\,dx = \ldots ?$$

$u = x^3+1$

$du = 3x^2\,dx$

$$\int x^2\sqrt{x^3+1}\,dx = \frac{1}{3}\int \sqrt{t}\,dt = \frac{1}{3}\,\frac{t^{3/2}}{\frac{3}{2}} =$$

$t = x^3+1$

$dt = 3x^2\,dx$

$$= \frac{2}{9}\,t^{3/2} + C$$

$$= \frac{2}{9}(x^3+1)^{3/2} + C$$

$$\cancel{*} = \frac{2}{9}(x^3+1)^{3/2}x^3 - \frac{2}{3}\int x^2(x^3+1)^{3/2}\,dx =$$

$$\int x^2(x^3+1)^{3/2}\,dx = \frac{1}{3}\int t^{3/2}\,dt = \frac{1}{3}\,\frac{(x^3+1)^{\frac{5}{2}}}{\frac{5}{2}} + C$$

$$t = x^3+1$$
$$dt = 3x^2\,dx$$

$$= \frac{2}{15}(x^3+1)^{\frac{5}{2}} + C$$

$$\int x^2(x^3+1)^{\frac{3}{2}}dx = \int \frac{1}{3}\cdot \underbrace{3\cdot x^2}_{dt}(x^3+1)^{\frac{3}{2}}\underbrace{dx}$$

$$= \frac{2}{9}(x^3+1)^{3/2}x^3 - \frac{4}{45}(x^3+1)^{\frac{5}{2}} + C =$$

$$= (x^3+1)^{3/2}\cdot\frac{2}{9}\Big(\underbrace{x^3 - \frac{2}{5}(x^3+1)}_{\frac{3}{5}x^3 - \frac{2}{5}}\Big) + C$$

INTEGRIRANJE Z RAZCEPOM NA PARCIALNE ULOMKE

$$\frac{f(x)}{g(x)}\ ;\ st f < st g$$

$$\frac{f(x)}{(x-a_1)^{m_1}(x-a_2)^{m_2}\cdots(x-a_k)^{m_k}(x^2+b_1x+c_1)^{n_1}\cdots(x^2+b_lx+c_l)^{n_l}} =$$

nerazcepni kvadratni faktorji

zapišemo kot vsoto ulomkov, člene dodajamo tako:

- za vsak linearen člen $(x-a_i)^{m_i}$ vsoti dodamo

$$\frac{A^i_1}{x-a_i} + \frac{A^i_2}{(x-a_i)^2} + \dots + \frac{A^i_{m_i}}{(x-a_i)^{m_i}}$$

- za vsak nerazcepen $(x^2+b_ix+c_i)^{n_i}$ dodamo

$$\frac{B^i_1x + C^i_1}{x^2+b_ix+c_i} + \dots + \frac{B^i_{m_i}x + C^i_{m_i}}{(x^2+b_i+c_i)^{m_i}}$$

Potem damo ulomke na skupni imenovalec, s primerjavo dobljenih koeficientov s tistimi v f dobimo konstante A^i_*, B^i_*, C^i_*.

$\frac{f}{g}$, $\operatorname{st} f \geq \operatorname{st} g \Rightarrow$ delimo f z g in računamo naprej

$$\int \frac{x-2}{(x+1)(x-1)^2}\,dx = \cancel{x}$$

$$\frac{x-2}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2} =$$

$$= \frac{A\underbrace{(x-1)^2}_{x^2-2x+1} + B\overbrace{(x+1)(x-1)}^{x^2-1} + C(x+1)}{(x+1)(x-1)^2}$$

$$= \frac{x^2(A+B) + x(-2A+C) + A-B+C}{(x+1)(x-1)^2}$$

$$= x-2$$

$$A + B = 0 \Rightarrow B = -A$$

$$-2A + C = 1 \Rightarrow C = 1 + 2A = 1 - 2\cdot\frac{3}{4} =$$

$$A - B + C = -2 \qquad = 1 - \frac{3}{2} = -\frac{1}{2}$$

$$A - B + C = A + A + 1 + 2A =$$

$$= 4A + 1 = -2$$

$$A = -\frac{3}{4}$$

$$B = \frac{3}{4}$$

$$C = -\frac{1}{2}$$

$$\ast = -\frac{3}{4}\int \frac{dx}{x+1} + \frac{3}{4}\int \frac{dx}{x-1} - \frac{1}{2}\int \frac{dx}{(x-1)^2}\,dx$$

$$= -\frac{3}{4}\ln(x+1) + \frac{3}{4}\ln|x-1| + \frac{1}{2}\,\frac{1}{x-1} + C$$

$$\int \frac{x^3+3}{x^2-2x-3}\,dx = \ast$$

$$(x^3+3) : (x^2-2x-3) = x+2$$

$$\begin{array}{l} -(x^3-2x^2-3x) \\ \hline 2x^2+3x+3 \\ -(2x^2-4x-6) \\ \hline 7x+9 \end{array}$$

$$x^3+3 = (x+2)(x^2-2x-3) + 7x+9$$

$$\ast \int (x+2) + \frac{7x+9}{x^2-2x-3}\,dx =$$

$$= \frac{(x+2)^2}{2} + \int \frac{7x+9}{x^2-2x-3}\,dx$$

$$\int \frac{7x+9}{x^2-2x-3}\,dx = \ast$$

$$\frac{7x+9}{x^2-2x-3} = \frac{7x+9}{(x+1)(x-3)} =$$

$$= \frac{A}{x+1} + \frac{B}{x-3}$$

"metoda pokritja / cover-up method)

$$\frac{7x+9}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3} \quad \Big| \, (x+1)$$

$$\frac{7x+9}{x-3} = A + \frac{B(x+1)}{x-3} \quad \Big| \, x=-1$$

$$\frac{2}{-4} = A \quad \ldots \quad A = -\frac{1}{2}$$

$$\frac{7x+9}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3} \quad \Big| \, (x-3)$$

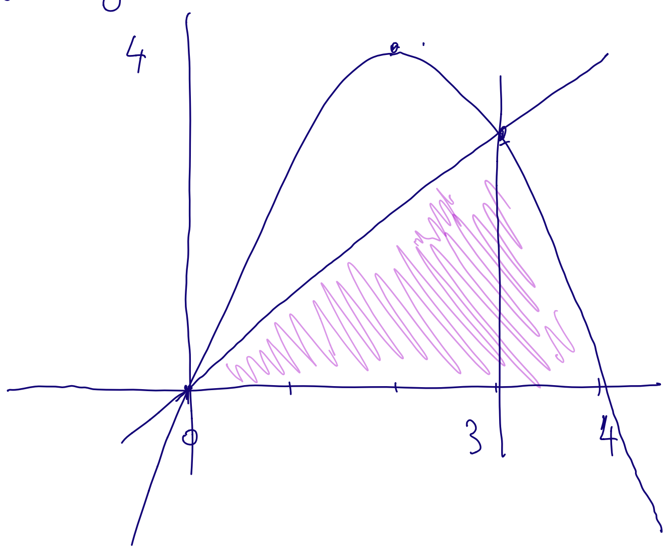
$$\frac{7x+9}{x+1} = \frac{A(x-3)}{x+1} + B \quad \Big| \, x=3$$

$$\frac{15}{2} = \frac{30}{4} = B$$

$\cancel{0} - \frac{1}{2}\int \frac{dx}{x+1} + \frac{15}{2}\int \frac{dx}{x-3} \ \ldots$

Računanje ploščin s pomočjo integrala

Izračunaj ploščino lika, ki ga omejujejo abscisna os, graf $f(x) = -x(x-4)$ in graf $g(x) = x$



$$-x(x-4) = x \quad | : x$$

$$4 - x = 1$$

$$\underline{\underline{x = 3}}$$

od $x = 2$ naprej je f strogo padajoča, g pa strogo naraščajoča funkcija.

ploščina: $\int^{3} g(x)\,dx + \int^{4} f(x)\,dx =$

0 3

$$= \int_0^3 x\,dx + \int_3^4 -x(x-4)\,dx =$$

$$= \frac{x^2}{2}\Big|_0^3 - \int_3^4 x^2 - 4x\,dx =$$

$$= \frac{9}{2} - \frac{x^3}{3}\Big|_3^4 + 4\,\frac{x^2}{2}\Big|_3^4 =$$

$$= \frac{9}{2} - \frac{64-27}{3} + 2\cdot(16-9) = \dots$$

INTEGRAL TRIGONOMETRIČNIH FUNKCIJ

$$\int \cos^5 x \,\sin^4 x\,dx = \int (1-t^2)^2\, t^4\,dt =$$

$$\cos^2 x = 1 - \sin^2 x$$

$$t = \sin x$$

$$dt = \cos x\,dx$$

$$\cos^4 x = \left[\cos^2 x\right]^2 = \left(1 - \sin^2 x\right)^2$$