NALOGA: POKAŽI, DA SE FUNKCIJI

$$f(x) = \operatorname{arctg} x \quad \text{IN} \quad g(x) = \frac{1}{2} \operatorname{arctg} \frac{2x}{1-x^2},$$

KJER STA DEFINIRANI, RAZLIKUJETA LE ZA KONSTANTO. SKICIRAJ JU!

REŠITEV:

KLJUČNA IDEJA JE POKAZATI, DA $f' = g'$ NA $D_f \cap D_g$. POTEM BO SLEDILO, DA JE $f = g + C$, $C \in \mathbb{R}$.

$$f'(x) = \frac{1}{1+x^2}$$

$$g'(x) = \frac{1}{2} \frac{1}{1+\left(\frac{2x}{1-x^2}\right)^2} \cdot \frac{2(1-x^2)+2x\cdot 2x}{(1-x^2)^2} =$$

$$= \frac{1}{\cancel{2}} \frac{\cancel{(1-x^2)^2}}{1-2x^2+x^4+4x^2} \frac{\cancel{2}+\cancel{2}x^2}{\cancel{(1-x^2)^2}} = \frac{1+x^2}{(1+x^2)^2} = \frac{1}{1+x^2}$$

ALI TO POMENI, DA JE $f(x) = g(x) + C$ NA CELI RAVNINI $\mathbb{R}$? NE, KER VELJA:

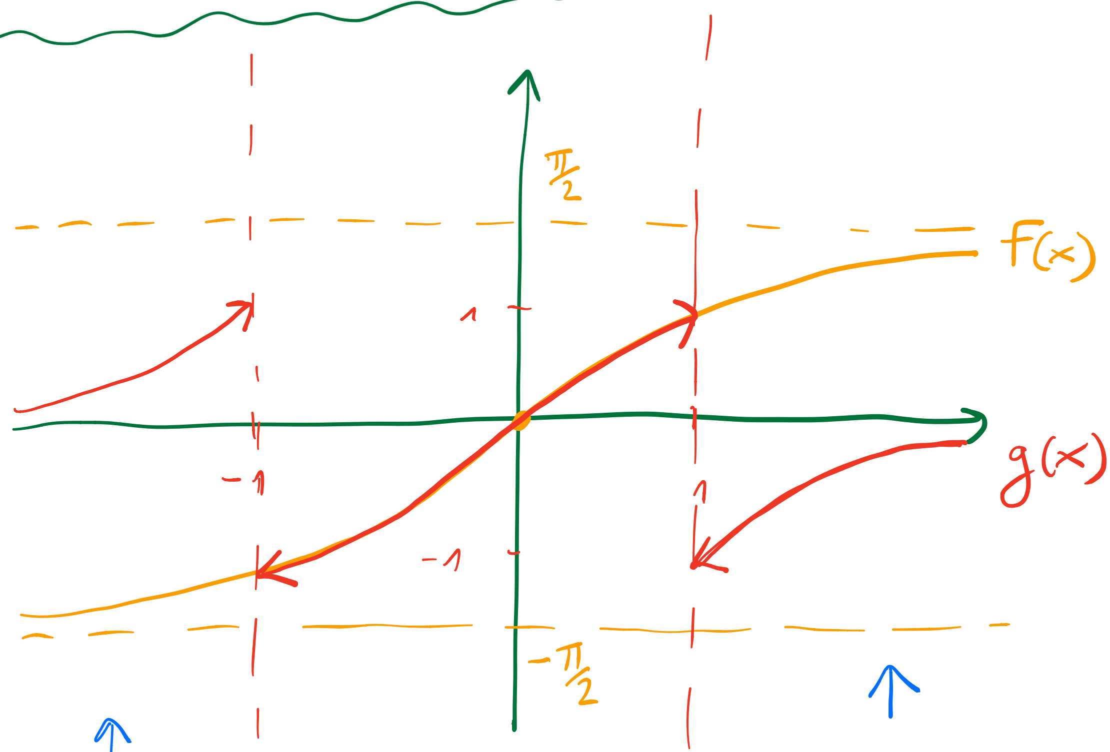
$$D_f = \mathbb{R} \quad \text{IN} \quad D_g = \mathbb{R} \setminus \{\pm 1\}.$$

ŠE VEČ, OPAZIMO TUDI:

- $f(0) = g(0) = \operatorname{arctg} 0 = 0$
- $\lim_{x\to\infty} f(x) = \frac{\pi}{2}$, $\quad \lim_{x\to\infty} g(x) = \lim_{x\to\infty} \operatorname{arctg} \cancelto{0}{\frac{2x}{1-x^2}} = 0$

$\lim\limits_{x\to-\infty} f(x) = -\frac{\pi}{2}$, $\lim\limits_{x\to-\infty} g(x) = \lim\limits_{x\to-\infty} \operatorname{arctg}\frac{2x}{1-x^2} = 0$

SKLEP: $f(x) = g(x) + C_j$ ZA $x \in \mathbb{R} \setminus \{\pm 1\}$, VENDAR PA SE KONSTANTA $C_j$ RAZLIKUJE GLEDE NA TO, ALI JE $x \in (-\infty, -1)$, $x \in (-1, 1)$ ALI $x \in (1, \infty)$.




$f = g - \frac{\pi}{2}$ | $f = g$ | $f = g + \frac{\pi}{2}$

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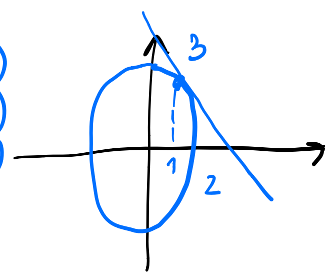
NALOGA: ZAPIŠI ENAČBO TANGENTE NA ELIPSO

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 \quad \text{V TOČKI } (1, c),\ c > 0.$$

SKICA: $a = 2$, $b = 3$

POLOSI



IZRAČUNAJMO $c > 0$:

$$\frac{x^2}{4}+\frac{y^2}{9}=1 \quad \Big/ \; x=1,\ c>0$$

$$\frac{1}{4}+\frac{c^2}{9}=1$$

$$\frac{c^2}{9}=\frac{3}{4}$$

$$c^2=\frac{27}{4} \Rightarrow c=\frac{3\sqrt{3}}{2}$$

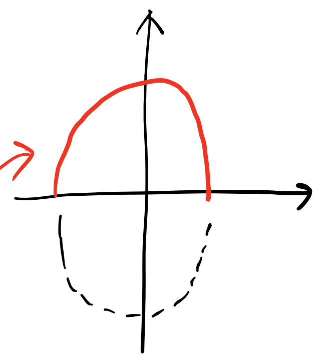
SEDAJ POTREBUJEMO SMERNI KOEFICIENT:

~ 1. MOŽNOST: EKSPLICITNO.

$$\frac{x^2}{4}+\frac{y^2}{9}=1$$

$$y^2=9\left(1-\frac{x^2}{4}\right)$$

$$y=\pm 3\sqrt{1-\frac{x^2}{4}}$$



IZBEREMO +, KER NAS ZANIMA ZGORNJI DEL ELIPSE.

$$k=y'(1)$$

IZRAČUNAJ SAM OZ. SAMA.

~ 2. MOŽNOST: IMPLICITNO ODVAJANJE

ENAČBO $\frac{x^2}{4}+\frac{y^2}{9}=1$ ODVAJAMO V TEJ OBLIKI IN UPOŠTEVAMO, DA JE $y=y(x)$.

$$\frac{x^2}{4}+\frac{y^2}{9}=1 \quad \Big/\,'$$

$$\frac{2x}{4}+\frac{2y\cdot y'}{9}=0$$

TA ČLEN SE POJAVI ZARADI VERIŽNEGA PRAVILA:

$$\left(y(x)^2\right)'=2y(x)\cdot y'(x).$$

SEDAJ VSTAVIMO $x=1$ IN $y(1)=\frac{3\sqrt{3}}{2}$:

$$\frac{1}{2}+\frac{2\cdot y(1)\cdot y'(1)}{9}=0$$

$$\frac{1}{2}+\frac{3\sqrt{3}\cdot y'(1)}{9}=0$$

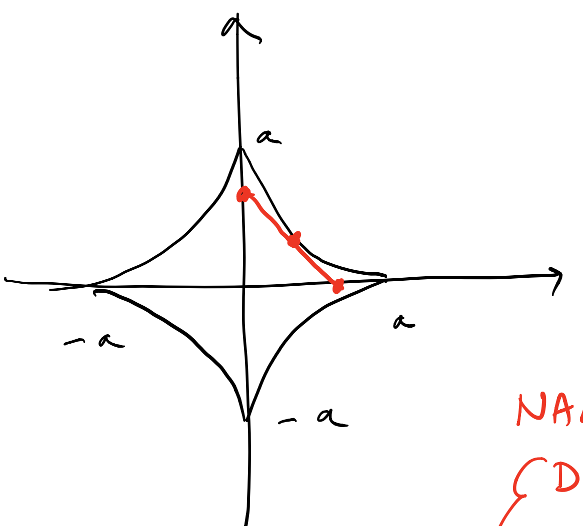
$$y'(1)=-\frac{3}{2\sqrt{3}}=-\frac{\sqrt{3}}{2}$$

ENAČBA TANGENTE:

$$y-\frac{3\sqrt{3}}{2}=-\frac{\sqrt{3}}{2}(x-1)$$

$$y=-\frac{\sqrt{3}}{2}x+2\sqrt{3}$$

PREVERI, DA NA PRVI NAČIN DOBIŠ ISTO ENAČBO.




ASTEROIDA:

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}},\ a>0$$

NALOGA:

DOKAŽI, DA JE DOLŽINA ODSEKA TANGENTE MED KOORDINATNIMA OSEMA KONSTANTNA.

REŠITEV: BREZ ŠKODE ZA SPLOŠNOST SE OMEJIMO NA PRVI KVADRANT t.j. POIŠČEMO ENAČBO TANGENTE V TOČKI $(x_0, y_0)$, $x_0 \geq 0$ IN $y_0 \geq 0$.

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} \quad /'$$

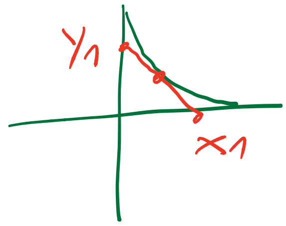
$$\cancel{\tfrac{2}{3}}\, x^{-\frac{1}{3}} + \cancel{\tfrac{2}{3}}\, y^{-\frac{1}{3}} \cdot y' = 0 \quad / \; x = x_0,\ y = y_0$$

$$y'(x_0) = -\sqrt[3]{\frac{y_0}{x_0}} = k_t, \quad x_0 \neq 0.$$

$x_0 = 0$ NA KONCU

ENAČBA:

$$y - y_0 = -\sqrt[3]{\frac{y_0}{x_0}}\,(x - x_0)$$




IZRAČUNAJMO PRESEČIŠČI $x_1$ Z ABSCISO IN $y_1$ Z ORDINATO.

$y = 0$:
$$-y_0 = -\sqrt[3]{\frac{y_0}{x_0}}\,(x_1 - x_0)$$

$$\sqrt[3]{x_0} \cdot \sqrt[3]{y_0^2} + x_0 = x_1$$

$$x_1 = \sqrt[3]{x_0}\left(\sqrt[3]{y_0^2} + \sqrt[3]{x_0^2}\right)$$

$$= \sqrt[3]{x_0}\left(x_0^{\frac{2}{3}} + y_0^{\frac{2}{3}}\right) = \sqrt[3]{x_0}\, a^{\frac{2}{3}}$$

KLJUČEN TRIK

KER $(x_0, y_0)$ NA ASTEROIDI.

$x = 0$:
$$y_1 - y_0 = -\sqrt[3]{\frac{y_0}{x_0}}\,(-x_0)$$

$$y_1 = y_0 + \sqrt[3]{y_0}\,\sqrt[3]{x_0^2}$$

$$y_1 = \sqrt[3]{y_0}\left(x_0^{\frac{2}{3}} + y_0^{\frac{2}{3}}\right) = \sqrt[3]{y_0}\cdot a^{\frac{2}{3}}$$

SEDAJ IZRAČUNAMO RAZDALJO:

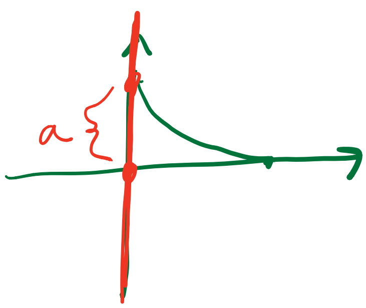
$$d\left((x_1,0),(0,y_1)\right) = \sqrt{x_1^2 + y_1^2} =$$

$$= \sqrt{x_0^{\frac{2}{3}} a^{\frac{4}{3}} + y_0^{\frac{2}{3}} a^{\frac{4}{3}}} = a^{\frac{2}{3}}\sqrt{x_0^{\frac{2}{3}} + y_0^{\frac{2}{3}}} =$$

$$= a^{\frac{2}{3}}\sqrt{a^{\frac{2}{3}}} = \underline{\underline{a}}$$

$(x_0, y_0)$ JE NA ASTEROIDI

DODATEK: IZPUSTILI SMO PRIMER, KO JE $x_0 = 0$. ENOSTAVNO JE PREVERITI, DA JE RAZDALJA TUDI TAM ENAKA $a$.




TANGENTA JE NAVPIČNA, ODSEK PA JE DOLG $a>0$.