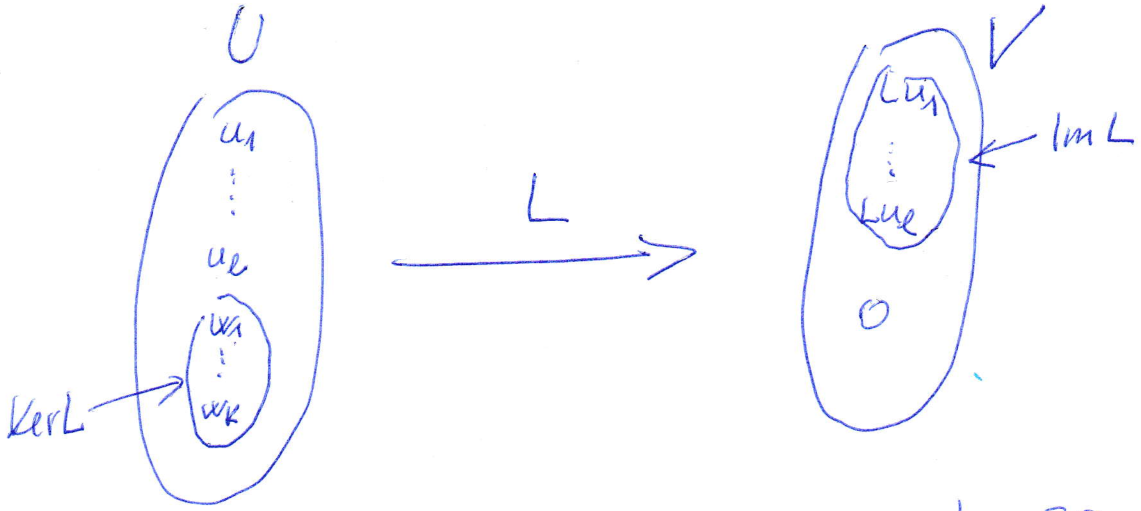

Trdimo, da je $Lu_1, \dots, Lu_l$ baza za $\operatorname{Im} L$.

Dokazati moramo, da je ta množica ogrodje in da je linearno neodvisna

(1) Dokažimo, da je $Lu_1, \dots, Lu_l$ ogrodje za $\operatorname{Im} L$

Vzemimo poljuben element $v \in \operatorname{Im} L$.

Po definiciji $\operatorname{Im} L$ obstaja tak element $u \in U$, da velja $v = Lu$

Element u razvijmo po bazi prostora U

$$u = \alpha_1 u_1 + \dots + \alpha_l u_l + \beta_1 w_1 + \dots + \beta_k w_k$$

Odtod sledi, da je

$$v = Lu = L(\alpha_1 u_1 + \dots + \alpha_l u_l + \beta_1 w_1 + \dots + \beta_k w_k)$$

$$\overset{\text{Lin}}{=} \alpha_1 Lu_1 + \dots + \alpha_l Lu_l + \beta_1 \underbrace{Lw_1}_{=0} + \dots + \beta_k \underbrace{Lw_k}_{=0}$$

$$= \alpha_1 Lu_1 + \dots + \alpha_l Lu_l$$

$$\Rightarrow v \in \operatorname{Lin}\{Lu_1, \dots, Lu_l\} \Rightarrow \operatorname{Im} L = \operatorname{Lin}\{Lu_1, \dots, Lu_l\}$$

② Dokažimo, da so $Lu_1, \ldots, Lu_\ell$ lin. neodvisni.

Pa recimo, da je

$$\gamma_1 Lu_1 + \ldots + \gamma_\ell Lu_\ell = 0$$

Radi bi dokazali, da je $\gamma_1 = \ldots = \gamma_\ell = 0$

Ker je L linearna preslikava, velja

$$L(\gamma_1 u_1 + \ldots + \gamma_\ell u_\ell) =$$

$$= \gamma_1 Lu_1 + \ldots + \gamma_\ell Lu_\ell = 0$$

Po definiciji $\ker L$, to pomeni, da

$$\gamma_1 u_1 + \ldots + \gamma_\ell u_\ell \in \ker L$$

Upoštevajmo, da je $w_1, \ldots, w_k$ baza za $\ker L$

Razvijmo $\gamma_1 u_1 + \ldots + \gamma_\ell u_\ell$ po tej bazi. Velja

$$\gamma_1 u_1 + \ldots + \gamma_\ell u_\ell = \delta_1 w_1 + \ldots + \delta_k w_k$$

za primerne $\delta_1, \ldots, \delta_k \in F$. Odtod sledi

$$\gamma_1 u_1 + \ldots + \gamma_\ell u_\ell + (-\delta_1) w_1 + \ldots + (-\delta_k) w_k = 0.$$

Upoštevajmo, da je $u_1, \ldots, u_\ell, w_1, \ldots, w_k$ baza U

Odtod sledi $\gamma_1 = \ldots = \gamma_\ell = -\delta_1 = \ldots = -\delta_k = 0$

Dokazali smo, da je res $\gamma_1 = \ldots = \gamma_\ell = 0$

Torej so $Lu_1, \ldots, Lu_\ell$ res linearno neodvisni

$B = \{u_1, \ldots, u_e, w_1, \ldots, w_k\}$

$\mathcal{C} = \{Lu_1, \ldots, Lu_e, z_1, \ldots, z_s\}$

$$Lu_1 = 1 \cdot Lu_1 + \ldots + 0 \cdot Lu_e + 0 \cdot z_1 + \ldots + 0 \cdot z_s$$

$$\vdots$$

$$Lu_e = 0 \cdot Lu_1 + \ldots + 1 \cdot Lu_e + 0 \cdot z_1 + \ldots + 0 \cdot z_s$$

$$Lw_1 = 0 \cdot Lu_1 + \ldots + 0 \cdot Lu_e + 0 \cdot z_1 + \ldots + 0 \cdot z_s$$

$$\vdots$$

$$Lw_k = 0 \cdot Lu_1 + \ldots + 0 \cdot Lu_e + 0 \cdot z_1 + \ldots + 0 \cdot z_s$$

$$[L]_{\mathcal{C} \leftarrow B} = \begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \\ 0 & & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix} \qquad v = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

$$Av = \begin{bmatrix} a_{1,1} \cdot \alpha_1 + \cdots + a_{1,n} \cdot \alpha_n \\ \vdots \\ a_{m,1} \cdot \alpha_1 + \cdots + a_{m,n} \cdot \alpha_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{1,1} \cdot \alpha_1 \\ \\ a_{m,1} \cdot \alpha_1 \end{bmatrix} + \cdots + \begin{bmatrix} a_{1,n} \cdot \alpha_n \\ \vdots \\ a_{m,n} \cdot \alpha_n \end{bmatrix}$$

$$= \alpha_1 \underbrace{\begin{bmatrix} a_{1,1} \\ \\ a_{m,1} \end{bmatrix}}_{\vec{a_1} = \text{ prvi stolpec } A} + \cdots + \alpha_n \underbrace{\begin{bmatrix} a_{1,n} \\ \vdots \\ a_{m,n} \end{bmatrix}}_{\vec{a_n} = \text{ n-ti stolpec } A}$$

$$\{Av \mid v \in F^n\} = \mathrm{Lin}\{\vec{a_1}, \ldots, \vec{a_n}\}$$

$$\{Av \mid v \in F^n\} = \mathrm{im} A$$

$$\mathrm{Lin}\{\vec{a_1}, \ldots, \vec{a_n}\} = \mathrm{Col}(A)$$

(stolpčni prostor A)