LINEARNA ALGEBRA 2020/21
16. VAJE: 17. 2. 2021

1. $B = \{(1,2,3,4)^T, (-2,1,0,1)^T, (1,0,1,1)^T, (2,-1,0,3)^T\}$ je baza $\mathbb{R}^4$. Izrazi standardno bazo $\mathbb{R}^4$ z B. Menjavo baze zapiši s prehodno matriko.

2. Prepričaj se, da bazi $B_1 = \{(1,0,-1,1)^T, (0,1,-1,1)^T\}$ in $B_2 = \{(2,-1,-1,1)^T, (1,1,-2,2)^T\}$ določata isti vektorski podprostor. Določi prehodni matriki med bazama.

3. Kdaj je »linearna« funkcija $f(x) = ax + b$ linearna preslikava?

4. Naj bo $\vec{a} = (a_1, a_2, a_3)(\cancel{\neq (1,2,0)}) \in R^3$. Naj bo $A(\vec{x}) = \vec{a} \times \vec{x}$. Preslikavi A določi pripadajočo matriko v standardni bazi.

   Naj bo $\vec{b} \neq 0$ in $\langle \vec{a}, \vec{b} \rangle = 0$. Pokaži, da je $\{\vec{a}, \vec{b}, \vec{a} \times \vec{b}\}$ baza $\mathbb{R}^3$. Preslikavi A določi pripadajočo matriko v tej bazi.

   $U \neq \{0\}$

5. Za linearno preslikavo $A\colon U \to V$ pokaži ekvivalenco naslednjih trditev.

   a) A je bijektivna

   b) Za vsako bazo $\{u_i \mid i = 1, \ldots, m\}$ prostora U je $\{Au_i \mid i = 1, \ldots, m\}$ baza prostora V.

   c) Obstaja baza $\{u_i \mid i = 1, \ldots, m\}$ prostora U, za katero je $\{Au_i \mid i = 1, \ldots, m\}$ baza prostora V.

Prehodne matrike

V - vektorski prostor $B = \{v_1, \dots, v_n\}$ dve bazi
$C = \{u_1, \dots, u_n\}$

$v \in V$ razvijeno po bazi

$$v = \sum_{i=1}^{n} \alpha_i v_i = \sum_{i=1}^{n} \beta_i u_i \qquad [v]_B = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \quad [v]_C = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}$$

Razvijeno v_i po bazi C

$$v_i = x_{1i} u_1 + x_{2i} u_2 + \dots + x_{ni} u_n \qquad [v_i]_B = e_i \quad [v_i]_C = \begin{bmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{ni} \end{bmatrix}$$

Velja $[\gamma z + \delta w]_B = \gamma [z]_B + \delta [w]_B$

$$[v]_C = [\alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_n v_n]_C = \alpha_1 [v_1]_C + \alpha_2 [v_2]_C + \dots + \alpha_n [v_n]_C$$

$$= \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & & x_{2n} \\ \vdots & & & \\ x_{n1} & \dots & & x_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} \rightarrow P_{C \leftarrow B} [v]_B = [v]_C$$

$$\parallel$$

$$P_{C \leftarrow B} [v]_B = [v]_C \qquad P_{B \leftarrow C} = (P_{C \leftarrow B})^{-1}$$

prehodna matrika

1. $B = \{\overset{v_1}{(1,2,3,4)^T}, \overset{v_2}{(-2,1,0,1)^T}, \overset{v_3}{(1,0,1,1)^T}, \overset{v_4}{(2,-1,0,3)^T}\}$ je baza $\mathbb{R}^4$. Izrazi standardno bazo $\mathbb{R}^4$ z B. Menjavo baze zapiši s prehodno matriko.

$e_1 \quad e_2 \quad e_3 \quad e_4$

Poznamo $P_{S \leftarrow B} = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}$

$$v_1 = 1e_1 + 2e_2 + 3e_3 + 4e_4$$

$$\begin{bmatrix} 1 & -2 & 1 & 2 \\ 2 & 1 & 0 & -1 \\ 3 & 0 & 1 & 0 \\ 4 & 1 & 1 & 3 \end{bmatrix} e_i = v_i$$

Izračunamo inverz

$$\begin{bmatrix} 1 & -2 & 1 & 2 \\ 2 & 1 & 0 & -1 \\ 3 & 0 & 1 & 0 \\ 4 & 1 & 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ -\frac{7}{8} & -1 & \frac{5}{8} & \frac{1}{4} \\ -\frac{3}{2} & -3 & \frac{5}{2} & 0 \\ \frac{1}{8} & 0 & -\frac{3}{8} & \frac{1}{4} \end{bmatrix}$$

$e_1 = \frac{1}{2} v_1 - \frac{7}{8} v_2 - \frac{3}{2} v_3 + \frac{1}{8} v_4 \qquad e_2 = v_1 - v_2 - 3v_3$

$e_3 = -\frac{1}{2} v_1 + \frac{5}{8} v_2 + \frac{5}{2} v_3 - \frac{3}{8} v_4 \qquad e_4 = \frac{1}{4} v_2 + \frac{1}{4} v_4$

2. Prepričaj se, da bazi $B_1 = \{(1,0,-1,1)^T, (0,1,-1,1)^T\}$ in $B_2 = \{(2,-1,-1,1)^T, (1,1,-2,2)^T\}$ določata isti vektorski podprostor. Določi prehodni matriki med bazama.

$\mathrm{lin}\, B_1 = \mathrm{lin}\, B_2$

$B_1 \cup B_2 \longrightarrow$ lin. neodvisno podmnožico B_1 ?

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ -1 & -1 & -1 & -2 \\ 1 & 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix} \longrightarrow B_1 \text{ je baza } \mathrm{lin}(B_1 \cup B_2)$$

$$x_1 v_1 + x_2 v_2 + y_1 u_1 + y_2 u_2 = 0$$

$x_1 = -2y_1 - y_2$

$x_2 = y_1 - y_2$

$y_1 = 1 \quad y_2 = 0$

$x_1 = -2$

$x_2 = 1$

$y_1 = 0 \quad y_2 = 1$

$x_1 = -1$

$x_2 = -1$

$u_1 \quad -2v_1 + v_2 = 0$

$u_1 = 2v_1 - v_2$

$-v_1 - v_2 + u_2 = 0$

$u_2 = v_1 + v_2$

$v \in \mathrm{Lin}\{B_1\}$

$$\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} [v]_{B_1} = [v]_{B_2}$$

$[v] \in \mathbb{R}^2$

stolpec z dvema koeficientoma

3. Kdaj je »linearna« funkcija $f(x) = ax + b$ linearna preslikava?

$f: \mathbb{R} \to \mathbb{R}$

- V, U vektorska prostora

$f: V \to U$ je linearna preslikava, če je:

- aditivna $f(x+y) = f(x) + f(y)$
- homogena $f(\alpha x) = \alpha f(x)$

(en pogoj: $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$

$f(x) = ax + b$ $\quad a, b \in \mathbb{R}$

$f: \mathbb{R} \to \mathbb{R}$

$b = 0 \iff f$ je linearna

Naj bo f linearna

$f(0) = b$

$\quad\|$

$f(0+0) = f(0) + f(0) = 2b$ $\Rightarrow b = 2b \rightsquigarrow b = 0$

Naj bo $b = 0$

$f(\alpha x) = a\alpha x = \alpha(ax) = \alpha f(x)$

$f(x+y) = a(x+y) = ax + ay = f(x) + f(y)$

4. Naj bo $\vec{a} = (a_1, a_2, a_3)(\cancel{\neq (0,0,0)}) \in R^3$. Naj bo $A(\vec{x}) = \vec{a} \times \vec{x}$. Preslikavi A določi pripadajočo matriko v standardni bazi.

Naj bo $\vec{b} \neq 0$ in $\langle \vec{a}, \vec{b} \rangle = 0$. Pokaži, da je $\{\vec{a}, \vec{b}, \vec{a} \times \vec{b}\}$ baza $\mathbb{R}^3$. Preslikavi A določi pripadajočo matriko v tej bazi.

$$A(\vec{x}+\vec{y}) = \vec{a} \times (\vec{x}+\vec{y}) = \vec{a} \times \vec{x} + \vec{a} \times \vec{y} = A(\vec{x}) + A(\vec{y})$$

$$A(\alpha \vec{x}) = \vec{a} \times \alpha \vec{x} = \alpha (\vec{a} \times \vec{x}) = \alpha A$$

$\mathcal{A} : U \to V$ $\qquad$ $B \to$ baza U, $B = \{u_1, \dots, u_n\}$ $\qquad$ $C \to$ baza V, $C = \{v_1, \dots, v_m\}$

$\mathcal{A}$ lin. preslikava

$$\underbrace{A_{B \leftarrow C}}_{\text{matrika}} [u]_B = [\mathcal{A} u]_C$$

$$\boxed{\mathcal{A} : U \to U \to \text{ponavadi želimo } A_{B \leftarrow B}}$$

$$A e_i = A [u_i]_B = [\mathcal{A} u_i]_C = \alpha_{1i} [v_1]_C + \alpha_{2i} [v_2]_C + \dots + \alpha_{mi} [v_m]_C$$

razvijeno po bazi

$A e_i \to$ i-ti stolpec

$$= \begin{bmatrix} \alpha_{1i} \\ \vdots \\ \alpha_{mi} \end{bmatrix}$$

$$A_{C \leftarrow B} = A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \ddots & & \vdots \\ \vdots & & & \\ \alpha_{m1} & \cdots & & \alpha_{mn} \end{bmatrix}$$

Slike baznih vektorjev razvijemo po bazi in koeficiente zapišemo v stolpce.

$$A\vec{e}_1 = (a_1, a_2, a_3) \times (1,0,0) = (0, a_3, -a_2)$$
$$A\vec{e}_2 = (a_1, a_2, a_3) \times (0,1,0) = (-a_3, 0, a_1)$$
$$A\vec{e}_3 = (a_1, a_2, a_3) \times (0,0,1) = (a_2, -a_1, 0)$$

$$A_S = A_{S\leftarrow S} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

$B =$

Naj bo $\vec{b} \neq 0$ in $\langle \vec{a}, \vec{b} \rangle = 0$. Pokaži, da je $\{\vec{a}, \vec{b}, \vec{a} \times \vec{b}\}$ baza $\mathbb{R}^3$. Preslikavi A določi pripadajočo matriko v tej bazi.

Predpostavim $\vec{a} \neq 0$

A_B

$$A\vec{a} = \vec{a} \times \vec{a} = 0 = 0 \cdot \vec{a} + 0\,\vec{b} + 0\,\vec{a}\times\vec{b}$$

$$A\vec{b} = \vec{a} \times \vec{b} = \quad 0$$

formula za dvojne vektorski produkt

$$A(\vec{a}\times\vec{b}) = \vec{a} \times (\vec{a}\times\vec{b}) = \underline{\langle \vec{a}, \vec{b} \rangle \cdot \vec{a}} - \langle \vec{a}, \vec{a} \rangle\, \vec{b}$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = \langle \vec{a}, \vec{c} \rangle\, \vec{b} - \langle \vec{a}, \vec{b} \rangle\, \vec{c}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\langle a, a \rangle \\ 0 & 1 & 0 \end{bmatrix} = A_B$$

$$A_S = P_{S\leftarrow B}\, A_B\, \underline{P_{B\leftarrow S}} \quad \checkmark$$

$$A_S = P^{-1} A_B P$$

5. Za linearno preslikavo $A\colon \overset{U\neq\{0\}}{U} \to V$ pokaži ekvivalenco naslednjih trditev.

a) A je bijektivna

b) Za vsako bazo $\{u_i \mid i = 1, \ldots, m\}$ prostora U je $\{Au_i \mid i = 1, \ldots, m\}$ baza prostora V.

c) Obstaja baza $\{u_i \mid i = 1, \ldots, m\}$ prostora U, za katero je $\{Au_i \mid i = 1, \ldots, m\}$ baza prostora V.

$B: U \to V$ je injektivna $\iff (Bx = 0 \Rightarrow x = 0)$

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a ⇒ b

Privzamemo, da je A bijektivna

Izberimo bazo U $\{u_1, \ldots, u_m\}$ $\qquad \{Au_1, \ldots, Au_m\}$ je baza V

Pokazati moramo:

$Au_1, \ldots, Au_m$ so lin neodvisni in razpenjajo V

Enačba $\sum_{i=1}^{m} \alpha_i A u_i$ ima samo eno rešitev $(\alpha_1, \ldots, \alpha_m) = (0, 0, \ldots, 0)$

$$\sum_{i=1}^{m} A\alpha_i u_i \overset{\text{linearnost}}{=} A\left(\sum_{i=1}^{m} \alpha_i u_i\right) = 0 \xrightarrow{\text{injektivnost}} \sum_{i=1}^{m} \alpha_i u_i = 0$$

$\overset{\{u_1, \ldots, u_m\} \text{ je baza}}{\Rightarrow} (\alpha_1, \ldots, \alpha_m) = (0, 0, \ldots, 0)$

lin neodvisnost

$Au_1, \ldots, Au_m$ razpenja V

A je surjektivna $\quad \forall v \in V \;\; \exists u \in U \quad Au = v$

u razvijemo po bazi

$$u = \beta_1 u_1 + \ldots + \beta_m u_m \qquad v = Au = A \sum_{i=1}^{m} \beta_i u_i = \sum_{i=1}^{m} \beta_i A u_i$$

razpenjajo

$b \Rightarrow c$

Za vsako bazo B prostora U je $AB = \{Ab \mid b \in B\}$ baza prostora V

Izberemo neko bazo B $\qquad$ AB baza V .

$c \Rightarrow a$

$\{u_1, \ldots, u_m\}$ baza U in $\{Au_1 \ldots Au_m\}$ baza V

injektivnost

Predpostavimo $Au = 0$ $\qquad$ pokazati moramo $u = 0$

$u = \beta_1 u_1 + \ldots + \beta_m u_m$ $\qquad$ $\sum_{i=1}^{m} \beta_i A u_i = 0$ $\Rightarrow$ $(\beta_1, \ldots, \beta_m) = (0, \ldots, 0)$

$\{Au_i \mid i = 1 \ldots m\}$ je baza

$u = 0u_1 + 0u_2 + \ldots + 0u_m = 0$

surjektivnost

Za vsak $v \in V$ iščemo $u \in U$ $\quad$ $Au = v$

$v = \sum_{i=1}^{m} \alpha_i A_i u_i = A_i \left(\sum_{i=1}^{n} \alpha_i u_i \right)$ $\qquad$ $u = \sum_{i=1}^{m} \alpha_i u_i$

$Au = v$