LINEARNA ALGEBRA 2019/20
15. VAJE: 11. 1. 2021

1. S primerom pokaži, da za vektorske prostore v splošnem ne velja $V \cap (U + W) = (V \cap U) + (V \cap W)$ in $V + (U \cap W) = (V + U) \cap (V + W)$.

2. Naj bosta
$$U = \text{Lin}\{(0, 0, 1, 2, 0)^T, (1, 1, 1, 2, 2)^T, (1, 0, 1, -1, 0)^T\}$$
in
$$V = \text{Lin}\{(1, 1, 0, 0, 0)^T, (2, 1, 0, 0, 2)^T, (2, 0, 2, 1, 2)^T\}.$$
Poišči kakšno bazo $U + V$ in $U \cap V$. Poišči kakšno bazo komplementa $U \cap V$.

3. Naj bo $U = \text{Lin}\{(1, 2, 2, 5, 1), (3, 6, 1, 2, 0), (2, 4, -1, 0, 1)\}$. Določi kakšen komplement vektorskega podprostora U v $\mathbb{R}^4$.

4. Naj bo $Q = \{p \in \mathbb{R}[x] \mid p(1) = 0\}$ in $S = \{p \in \mathbb{R}[x] \mid p(2) = 0\}$. Določi $Q + S$ in $Q \cap S$. Naj bo $V_n = V \cap \mathbb{R}_n[x]$. Poišči kakšno bazo S_n, Q_n in $Q_n \cap S_n$. in $Q_n + S_n$.

5. Naj bo $U = V \oplus W$ in naj bo $V \subset Z \subset U$. Pokaži, da je $Z = V \oplus (Z \cap W)$.

6. Naj bo W vektorski prostor in V,V' in V'' vektorski podprostori, za katere velja
$$W = V \oplus V' = V \oplus V''$$
Pokaži, da je $\dim(V' \cap V'') \geq \dim W - 2 \dim V$.

7. Naj bo $(G, +)$ abelova grupa. Pokaži, da je na G možno vpeljati zunanjo operacijo, da bo G vektorski prostor nad $\mathbb{Q}$ natanko tedaj ko je preslikava $f_n \colon G \to G$ definirana z $f_n(g) = ng$ bijektivna.

1. S primerom pokaži, da za vektorske prostore v splošnem ne velja $V \cap (U + W) = (V \cap U) + (V \cap W)$ in $V + (U \cap W) = (V + U) \cap (V + W)$.

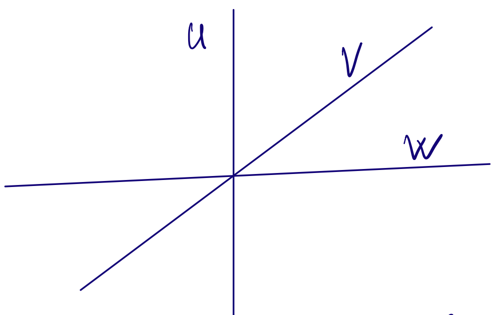
v splošnem

- $V \cap (U+W) \neq (V \cap U) + (V \cap W)$

$\mathbb{R}^2$

V, U, W – različne premice

$U + W = \underline{\mathbb{R}^2}$ – Leva stran



$V \cap \mathbb{R}^2 = \underline{V}$

$V \cap U = \{0\}$

$V \cap W = \{0\}$

$\underline{\{0\} + \{0\} = \{0\}}$

desna stran

$V \neq \{0\}$

v splošnem

- $V + (U \cap W) \neq (V \cap U) + (V \cap W)$

$\mathbb{R}$

$V = \mathbb{R}$ $\quad U = W = \{0\}$

$\mathbb{R} \neq \{0\} + \{0\} = \{0\}$

2. Naj bosta

$$U = \mathrm{Lin}\{\overset{u_1}{(0,0,1,2,0)^T}, \overset{u_2}{(1,1,1,2,2)^T}, \overset{u_3}{(1,0,1,-1,0)^T}\}$$

in

$$V = \mathrm{Lin}\{\overset{v_1}{(1,1,0,0,0)^T}, \overset{v_2}{(2,1,0,0,2)^T}, \overset{v_3}{(2,0,2,1,2)^T}\}.$$

Poišči kakšno bazo $U+V$ in $U\cap V$. Poišči kakšno bazo komplementa $U\cap V$.

$$\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = -\beta_1 v_1 - \beta_2 v_2 - \beta_3 v_3$$

$$-2\ \begin{bmatrix} 0 & 1 & 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 2 \\ 2 & 2 & -1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 2 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 & 2 & 2 \\ 0 & 0 & -3 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & -2 & 0 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 2 \\ & 1 & 0 & 1 & 1 & 0 \\ & & 1 & 0 & 0 & 1 \\ & & 0 & 0 & 1 & 1 \\ & & & 1 & 0 & -1 \end{bmatrix}$$

$$\begin{array}{cccccc} \alpha_1 & \alpha_2 & \alpha_3 & \beta_1 & \beta_2 & \beta_3 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 1 \\ & & & 1 & 0 & -1 \\ & & & & 1 & 1 \end{bmatrix}$$

u_1, u_2, u_3, v_1, v_2 - so baza $U+V$

$$\alpha_1 = -\beta_3$$
$$\alpha_2 = 0$$
$$\alpha_3 = -\beta_3$$
$$\beta_1 = \beta_3$$
$$\beta_2 = -\beta_3$$

$$\alpha_1 + \beta_3 = 0$$

$$\underbrace{-\beta_3 u_1}_{\alpha_1} + \underbrace{0 u_2}_{\alpha_2} \underbrace{- \beta_3 u_3}_{\alpha_3} + \underbrace{\beta_3 v_1}_{\beta_1} \underbrace{- \beta_3 v_2}_{\beta_2} + \underbrace{\beta_3 v_3}_{\beta_3} = 0$$

$$\beta_3(u_1 + u_3) = \beta_3(v_1 - v_2 + v_3) \rightsquigarrow$$

Presek je $\mathrm{Lin}\{v_1 - v_2 + v_3\}$

2.2.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & 0 & -1 & 0 \\ 1 & 0 & 2 & 1 & 1 & 1 \end{array}\right]$$

Naj bo
$U = \mathrm{Lin}\{u_1, u_2, u_3\}$
$V = \mathrm{Lin}\{v_1, v_2, v_3\}$
$= [u_1, u_2, u_3, v_1, v_2, v_3]$
Poišči bazo $U \cap V$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 & 1 & 2 \\ 0 & -1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 2 \\ & 1 & 1 & 1 & 1 & 2 \\ & & 1 & 1 & 0 & 1 \\ & & 0 & -1 & 1 & -1 \\ & & 0 & -2 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 2 \\ & 1 & 1 & 1 & 1 & 2 \\ & & 1 & 1 & 0 & 1 \\ & & & 1 & -1 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 1 & 1 \\ & & 1 & 0 & 1 & 0 \\ & & & 1 & -1 & 1 \end{bmatrix}$$

$\alpha_1 = 0$
$\alpha_2 = -\beta_2 - \beta_3$
$\alpha_3 = -\beta_2$
$\beta_1 = \beta_2 - \beta_3$

- $(-\beta_2 - \beta_3)u_2 - \beta_2 u_3 + (\beta_2 - \beta_3)v_1 + \beta_2 v_2 + \beta_3 v_3 = 0$

$\beta_2 = 1 \quad \beta_3 = 0$

$\beta_2 = 0 \quad \beta_3 = 1$

$(\beta_2, \beta_3) = \beta_2(1,0) + \beta_3(0,1)$

$\underline{u_2} + u_3 = \underline{v_1 + v_2}$

$u_2 = -v_1 + v_3$

$\Rightarrow \text{Lin}\{v_1 + v_2, -v_1 + v_3\}$

$\parallel$

$\text{Lin}\{u_2 + u_3, u_2\}$

3. Naj bo $U = \text{Lin}\{(1,2,2,5,1),(3,6,1,2,0),(2,4,-1,0,1)\}$. Določi kakšen komplement vektorskega podprostora U v $\mathbb{R}^4$.

$U \subseteq V$ komplement v. podprostora U je $W \subseteq V$, da je $U \oplus W = V$

notranja direktna vsota

$\oplus$ – direktna vsota

$U + W = V$ in $U \cap W = \{0\}$

U, W $\quad$ $\underline{U \oplus W} = \{(u, w) \mid u \in U, w \in W\}$

poljubna v.p. $\quad$ zunanja dir. vsota

$U_1 = \text{"}\{(u,0) \mid u \in U\} \subseteq U \oplus W$

$W_1 = \text{"}\{(0,w) \mid w \in W\} \subseteq U \oplus W$

Izberemo bazo U in jo dopolnimo do baze celotnega prostora. Dodani elementi so baza komplementa.

$$\begin{array}{c} \\ -2 \\ \\ \\ \\ \end{array}\begin{bmatrix} 1 & 3 & 2 & 1 & 0 & 0 & & \\ 2 & 6 & 4 & 0 & 1 & 0 & & \\ 2 & 1 & -1 & & & 1 & & \\ 5 & 2 & 0 & & & & 1 & \\ 1 & 0 & 1 & & & & & 1 \end{bmatrix}$$

Prvih 5 vektorjev je baza $\mathbb{R}^5$. 3., 4. stolpec dodana.

$$\begin{bmatrix} 1 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & \underline{-5} & -5 & \underline{-2} & 0 & 1 & 0 & 0 \\ 0 & \underline{-13} & -10 & -5 & 0 & 0 & 1 & 0 \\ 0 & \underline{-3} & -1 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & 5 & 2 & 0 & -1 & 0 & 0 \\ 0 & -3 & 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & -2 & 1 & 0 & 0 & 1 \end{bmatrix}\begin{array}{c} \\ \\ \\ -1 \\ \\ \end{array}$$

$$\begin{bmatrix} 1 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{5} & 0 & -\frac{1}{5} & 0 & 0 \\ \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 3 & \frac{1}{5} & 0 & -\frac{11}{5} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 & 1 & -1 \\ 0 & 0 & 0 & 2 & -1 & 0 & 0 & -1 \end{bmatrix} \quad \cdot(-3)$$

$$\begin{bmatrix} 1 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{5} & 0 & -\frac{1}{5} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 19 & -10 & 15 \\ 0 & 0 & 0 & 2 & -1 & 0 & 0 & -1 \end{bmatrix} \quad \cdot(-2)$$

$$0 \quad -1 \quad -38 \quad +20 \quad -31$$

$\mathrm{Lin}\{e_1, e_2\}$ je komplement U

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

4. Naj bo $Q = \{p \in \mathbb{R}[x] \mid p(1) = 0\}$ in $S = \{p \in \mathbb{R}[x] \mid p(2) = 0\}$. Določi $Q + S$ in $Q \cap S$. Naj bo $V_n = V \cap \mathbb{R}_n[x]$. Poišči kakšno bazo S_n, Q_n in $Q_n \cap S_n$. $Q_n + S_n$

$$Q + S = \mathbb{R}[x]$$

Poljuben polinom $r \in \mathbb{R}[x]$ lahko zapišemo $r(x) = q(x) + s(x)$, kjer je $q(x) \in Q$ in $s(x) \in S$

$a(x)$, $b(x)$ tuja polinoma potem obstaja polinoma $f(x), g(x)$, da je $a(x) f(x) + b(x) g(x) = 1$ (Evklidov algoritem)

$(x-1)$ $(x-2)$ sta tuja

$$(x-1) f(x) + (x-2) g(x) = 1$$

$$\underbrace{(x-1) f(x) \cdot r(x)}_{\in Q} + \underbrace{(x-2) g(x) \cdot r(x)}_{\in S} = r(x)$$

$$Q \cap S = \{ p \mid p(1) = p(2) = 0 \}$$

$$= \{ (x-1)(x-2) p \mid p \in \mathbb{R}[x] \}$$

$$Q_n = \{ p \in \mathbb{R}_n[x] \mid p(1)=0 \}$$

$$S_n = \{ p \in \mathbb{R}_n[x] \mid p(2)=0 \}$$

Baza Q_n S_n

$(x-1)\cdot p(x)$ stp $\leq n-1$

baza Q_n

$$\{(x-1), (x-1)x, (x-1)x^2 \dots (x-1)x^{n-1}\}$$

$$\sum_{i=0}^{n-1} \alpha_i (x-1)x^i \overset{\text{enakost polinomov}}{=} 0 \Rightarrow \sum_{i=0}^{n-1} \alpha_i x^i = 0 \Rightarrow \text{so lin neodvisni}$$

$(x-1)p(x)$ stp $\leq n-1$ p je lin kombinacija $1, x, \dots x^{n-1}$

Baza S_n

$$(x-2), (x-2)x, \dots (x-2)x^{n-1}$$

$$Q_n \cap S_n = \{ p \in \mathbb{R}_n[x] / p(1)=p(2)=0 \}$$

Baza $Q_n \cap S_n$

$$(x-1)(x-2), (x-1)(x-2)x^1 \dots (x-1)(x-2)x^{n-2}$$

$Q_n + S_n \neq (Q + S) \cap \mathbb{R}_n[x]$

$\subset$ – vsebovanost vedno velja

$Q_n \leq Q + S \quad Q_n \subseteq \mathbb{R}_n[x]$

$S_n \quad -\parallel- \quad S_n \subseteq -\parallel-$

$$\dim(Q_n + S_n) = \underbrace{\dim(Q_n)}_{= n} + \underbrace{\dim(S_n)}_{= n} - \underbrace{\dim(Q_n \cap S_n)}_{= n-1}$$

Razen, če je $n = 0$

- $n > 0 \qquad \dim(Q_n + S_n) = n+1 = \dim(\mathbb{R}_n[x]) \Rightarrow Q_n + R_n = \mathbb{R}_n[x]$

- $n = 0 \qquad Q_0 + S_0 = \{0\}$