Primeri uporabe napak

Naloga: Rekurzivno želimo računati

člene zaporedij:

a) $a_{n+1} - \frac{5}{2} a_n + a_{n-1} = 0, \quad a_0 = 1, \ a_1 = \frac{1}{2},$

b) $a_{n+1} - \frac{10}{3} a_n + a_{n-1} = 0, \quad a_0 = 1, \ a_1 = \frac{1}{3}.$

a) $a_n = \lambda^n; \ \lambda \in \mathbb{R}$

$$\boxed{\lambda^{n+1} - \frac{5}{2}\lambda^n + \lambda^{n-1} = 0} \quad \lambda \neq 0$$

$$\lambda^2 - \frac{5}{2}\lambda + 1 = 0$$

$$(\lambda - 2)(\lambda - \tfrac{1}{2}) = 0$$

$$\lambda_1 = 2, \quad \lambda_2 = \frac{1}{2}$$

Če sta rešitvi $\lambda_1, \lambda_2 \in \mathbb{R}$ in $\lambda_1 \neq \lambda_2$:

$$a_n = \alpha \lambda_1^n + \beta \lambda_2^n$$

$$= \alpha 2^n + \beta 2^{-n}$$

Koef. α in β določimo iz začetnih pogojev:

$$a_0 = 1 = \alpha \cdot 2^0 + \beta \cdot 2^0$$

$$a_1 = \frac{1}{2} = \alpha \cdot 2 + \beta \cdot 2^{-1}$$

$\Rightarrow$ $\alpha = 0$ in $\beta = 1$ ✓

$$\boxed{a_n = 2^{-n}}$$

b) Podobno kot v a) dobimo karakt. enačbo

$$\lambda^2 - \frac{10}{3}\lambda + 1 = 0$$

$$(\lambda - 3)(\lambda - \tfrac{1}{3}) = 0$$

$$\lambda_1 = 3, \quad \lambda_2 = 1/3$$

Splošni člen je $a_n = \gamma 3^n + \delta 3^{-n}$. Iz $a_0 = 1$, $a_1 = 1/3$ dobimo, da je $\gamma = 0$ in $\delta = 1$, torej

$$a_n = 3^{-n}$$

Dobimo katastrofalne napake! Zakaj?

V resnici rešujemo

$$b_{n+1} - \frac{10}{3} b_n + b_{n-1} = 0, \quad b_0 = 1, \quad b_1 = \frac{1}{3} + \varepsilon, \quad |\varepsilon| \le u \approx 10^{-16}$$

zanemarimo vpliv te napake!

Podobno kot prej:

$$\lambda^2 - \frac{10}{3}\lambda + 1 = 0$$

$$\lambda_1 = 3, \quad \lambda_2 = \frac{1}{3}$$

Iz začetnih pogojev sledi:

$$b_0 = 1 = A\, 3^0 + B \cdot 3^0 = A + B$$

$$b_1 = \frac{1}{3} + \varepsilon = A \cdot 3 + B \cdot 3^{-1} = 3A + \frac{1}{3} B$$

$$\Rightarrow \quad B = 1 - A \quad \text{in} \quad 3A + \frac{1}{3}(\cancel{1} - A) = \cancel{\frac{1}{3}} + \varepsilon$$

$$\frac{8A}{3} = \varepsilon$$

$$A = \frac{3 \cdot \varepsilon}{8}$$

$$B = 1 - A = \frac{8 - 3 \cdot \varepsilon}{8}$$

$$b_n = \frac{3\varepsilon}{8} 3^n + \frac{8-3\varepsilon}{8} 3^{-n}$$

Ko n raste, začne $\frac{3\varepsilon}{8} 3^n$ prevladovati