Numerično reševanje sistemov linearnih enačb

Množenje matrike in vektorja:

$A \in \mathbb{R}^{m \times n}, \; x \in \mathbb{R}^{n \times 1} \cong \mathbb{R}^n$

$Ax = ?$

Standardno: $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix},$

$x = [x_1, x_2, \dots, x_n]^T$

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{bmatrix}.$$

Alternativa:

$A = [a_1, a_2, \dots, a_n]; \quad a_j \in \mathbb{R}^m, \; j = 1, 2, \dots, n.$

$$Ax = x_1 a_1 + x_2 a_2 + \dots + x_n a_n = \sum_{j=1}^{n} x_j a_j \;!!!$$

Primer „uporabe":

Sistem $Ax = b$ ima rešitev natanko tedaj, ko je $\text{rang}(A) = \text{rang}([A, b])$.

($\Rightarrow$) Če ima $Ax = b$ rešitev, potem je

$$\sum_{j=1}^{n} x_j a_j = b,$$ kar pomeni, da je

b linearna komb. stolpcev $a_1, \dots, a_n$, torej je $\text{rang}(A) = \text{rang}([A, b])$.

($\Leftarrow$) Če je $\text{rang}(A) = \text{rang}([A, b])$, potem

$$b = \sum_{j=1}^{n} x_j a_j, \text{ za neke } x_j \in \mathbb{R}, \ j = 1, 2, \dots, n.$$

Če pišemo $x = [x_1, \dots, x_j]^T$, je

$$Ax = b$$ in sistem ima rešitev.

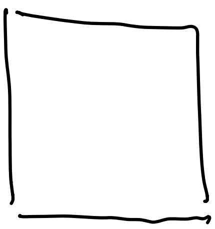
Matrični zapis sistema enačb:

$$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = b_1$$
$$\vdots$$
$$a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n = b_m$$

Matrično: $Ax=b$, $A=\begin{bmatrix} a_{11} & \dots & a_{1n}\\ \vdots & & \\ a_{m1} & \dots & a_{mn}\end{bmatrix} \in \mathbb{R}^{m\times n}$,

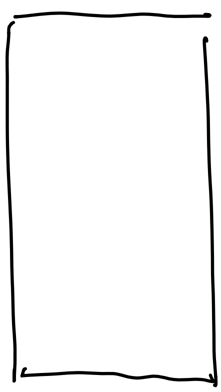
$x=[x_1,\dots,x_n]^T$, $b=[b_1,\dots,b_m]^T$.

Najprej se omejimo na primer $\underline{m=n}$!



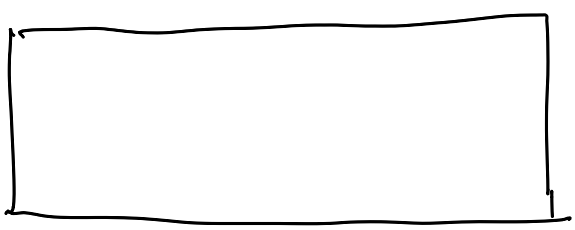
– kvadratni sistem

$m=n$



– predoločen sistem

$m>n$



– nedoločen sistem

$m<n$

Pomembni so sistemi, ki imajo posebno matriko:

① A zgornje trikotna:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ 0 & 0 & \cdots \; 0 & a_{nn} \end{bmatrix}.$$

(2) A spodnja trikotna

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

(3) A je tridiagonalna

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdot\, 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & a_{n,n-1} & a_{n,n} \end{bmatrix}.$$

(4) A je diagonalna

$$A = \begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{bmatrix} = \operatorname{diag}(a_{11}, \ldots, a_{nn}).$$

Vektorske in matrične norme

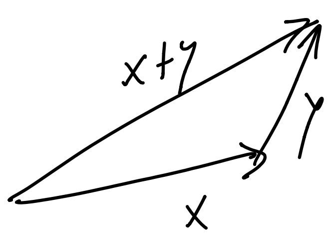
Definicija:

Vektorska norma je preslikava $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}^+ \cup \{0\}$, za katero je

① $\|x\| = 0 \iff x = 0$,

② $\|\alpha x\| = |\alpha| \, \|x\|$; $\alpha \in \mathbb{R}$,

③ $\|x+y\| \leq \|x\| + \|y\|$; $x, y \in \mathbb{R}^n$



Primeri:

① $\sqrt{x^T x} = \sqrt{\sum_{i=1}^{n} |x_i|^2} =: \|x\|_2$; 2-norma

② $\|x\|_1 = \sum_{i=1}^{n} |x_i|$, 1-norma

③ $\|x\|_p = \sqrt[p]{\sum^{n} |x_i|^p}$; p-norma ; $p \geq 1$

$i=1$

4. $\|x\|_\infty = \max_{i} |x_i|$; ∞-norma

<u>Definicija matrične norme:</u>

Matrična norma je preslikava $\|\cdot\|: \mathbb{R}^{n\times n} \to \mathbb{R}^+ \cup \{0\}$, za katero je:

1. $\|A\| = 0 \iff A = 0$,
2. $\|\alpha A\| = |\alpha| \, \|A\|$,
3. $\|A+B\| \leq \|A\| + \|B\|$,
4. submultiplikativnost:
   $\|AC\| \leq \|A\| \, \|C\|$

Najbolj znane matrične norme:

- $\|A\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|$ (1-norma)
- $\|A\|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|$ (∞-norma)

- $\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2}$ (Frobeniusova)

- $\|A\|_2 = \max_{1 \le i \le n} \sqrt{\lambda_i(A^T A)}$ (spektralna)

Splošno: Operatorska norma, ki je porojena z neko vektorsko normo $\|\cdot\|$, je

$$\|A\| := \sup_{\|x\|=1} \|Ax\|$$

Primer:
$$\sup_{\|x\|_2=1} \|Ax\|_2 =: \|A\|_2$$

Def.: Število občutljivosti (pogojenostno število) za matriko $A \in \mathbb{R}^{n \times n}$, $\det A \neq 0$, je

$$\mathcal{K}(A) = \|A^{-1}\| \, \|A\| .$$

Zanima nas, kako je rešitev sistema $Ax = b$ občutljiva na motnje v A in b.

Lema: Naj bo $Ax=b$ in $(A+\delta A)(x+\delta x)=b+\delta b$.

Če je $\|A^{-1}\|\,\|\delta A\|<1$, je

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{\kappa(A)}{1-\kappa(A)\dfrac{\|\delta A\|}{\|A\|}}\left(\frac{\|\delta A\|}{\|A\|}+\frac{\|\delta b\|}{\|b\|}\right).$$

Primer: $A_n = \left(\dfrac{1}{i+j}\right)_{i,j=1}^{n}$ (Hilbertova matrika).

Reševanje sistema $Lx=b$, kjer je L spodnja trikotna matrika $(\det L\neq 0)$.

$$L=\begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ \vdots & & & & \\ l_{n1} & l_{n2} & & \cdots & l_{nn} \end{bmatrix}, \quad L=\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}, \quad x=\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

$$\begin{aligned} l_{11}x_1 &= b_1 \\ l_{21}x_1 + l_{22}x_2 &= b_2 \\ &\vdots \\ l_{n1}x_1 + l_{n2}x_2 + \dots + l_{nn}x_n &= b_n \end{aligned}$$

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Direktno vstavljanje (prema substitucija)!

$$x_i = \frac{1}{l_{ii}}\left(b_i - \sum_{j=1}^{i-1} l_{ij}\, x_j\right), \quad i = 1, 2, \ldots, n.$$

Število operacij:

$$\sum_{i=1}^{n}\left(\overset{/}{\cancel{1}} \overset{-}{+1} + \overset{*}{i - \cancel{1}} + \overset{+}{i-2}\right) = \sum_{i=1}^{n} 2i - 1$$

$$= 2\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 = \cancel{2}\,\frac{n(n+1)}{\cancel{2}} - n$$

$$= n^2 + n - n = \underline{\underline{n^2}}$$