KRIVULJE V POLARNIH KOORDINATAH

OBRAVNAVALI BOMO RAVNINSKE KRIVULJE, KI SO PODANE Z ZVEZO:

$$r = r(\varphi), \quad r \ldots \text{ POLARNI RADIJ}$$
$$\varphi \in [\alpha, \beta] \quad \varphi \ldots \text{ POLARNI KOT}$$

RAZUMEMO JIH LAHKO KOT POSEBNO OBLIKO PARAMETRIČNIH KRIVULJ, KI SO PODANE S PARAMETROM φ:

$$\big(x(\varphi), y(\varphi)\big) = \big(r(\varphi)\cos\varphi, r(\varphi)\sin\varphi\big), \quad \varphi \in [\alpha, \beta].$$

PRI NAČRTOVANJU TEH KRIVULJ SI PRAVILOMA POMAGAMO KAR S TABELIRANJEM VREDNOSTI ZA φ IN r, TER SKICO ODVISNOSTI $r = r(\varphi)$.

NALOGA: NAJ BOSTA $a, b > 0$. SKICIRAJ KRIVULJE:

(a) $r(\varphi) = a(1+\cos\varphi), \quad \varphi \in [0, 2\pi]$.

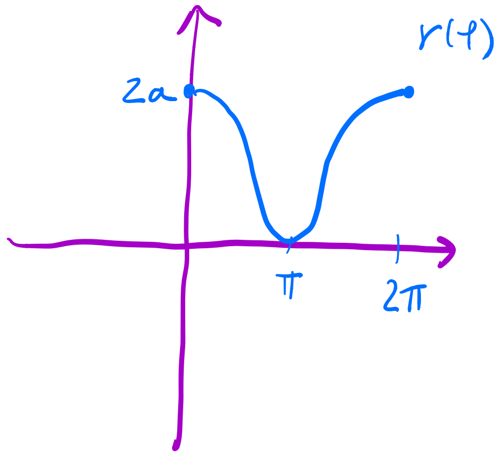
(b) $r(\varphi) = a\cos(3\varphi), \quad \varphi \in [0, 2\pi]$.

(c) $r(\varphi) = a + b\varphi, \quad \varphi \in [0, \infty)$

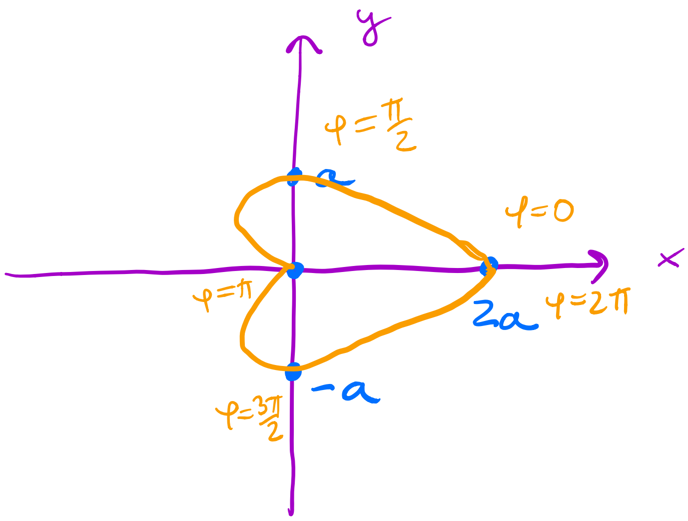
(d) $r(\varphi) = \dfrac{a}{\varphi}, \quad \varphi \in (0, \infty)$

REŠITEV:

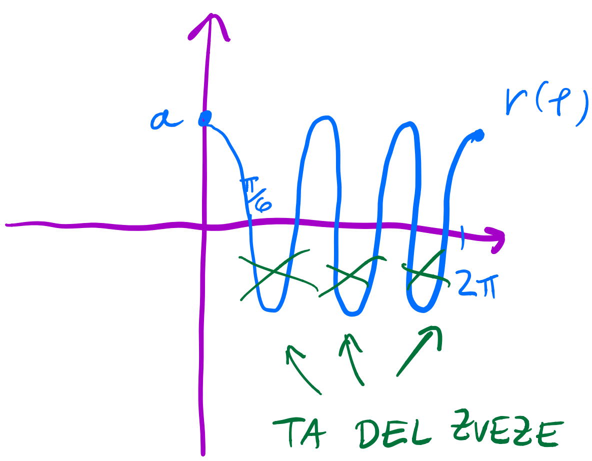
(a) $r(\varphi) = a(1+\cos\varphi)$, $\varphi \in [0, 2\pi]$.



φ	r
0	$2a$
$\frac{\pi}{2}$	a
π	0
$\frac{3\pi}{2}$	a
2π	$2a$

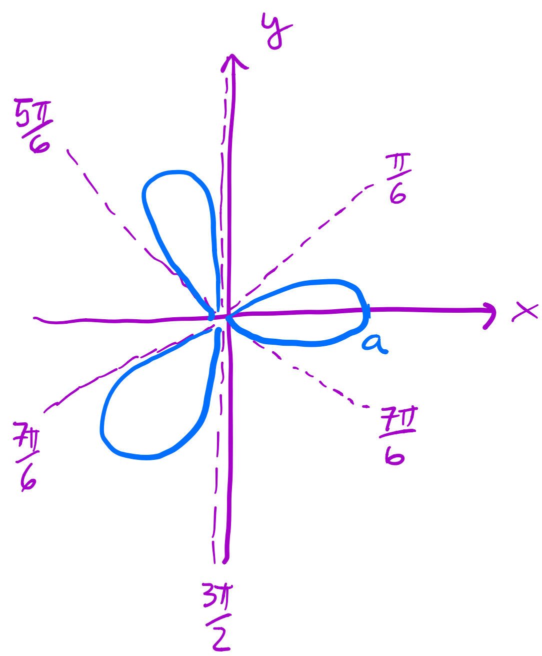


(b) $r(\varphi) = a\cos(3\varphi)$

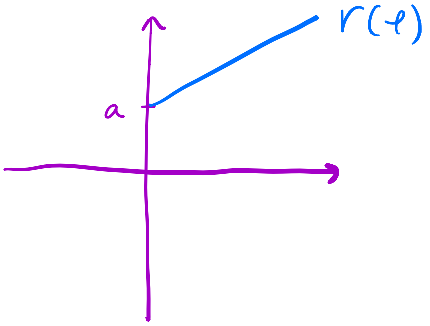


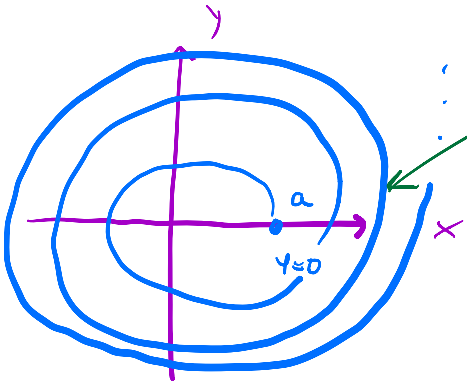
TA DEL ZVEZE ODPADE, KER $r \geq 0$.

φ	r
0	a
$\frac{\pi}{6}$	0
$\frac{\pi}{2}$	0
$\frac{2\pi}{3}$	a
$\frac{5\pi}{6}$	0
$\frac{7\pi}{6}$	0
$\frac{4\pi}{3}$	a
$\frac{3\pi}{2}$	0
$\frac{11\pi}{6}$	0
2π	a



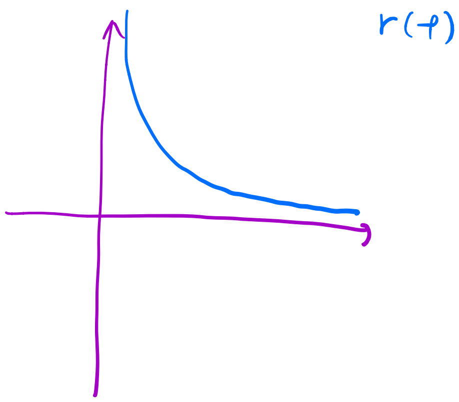
(c) $r(\varphi) = a + b\varphi, \quad \varphi \in [0, \infty)$

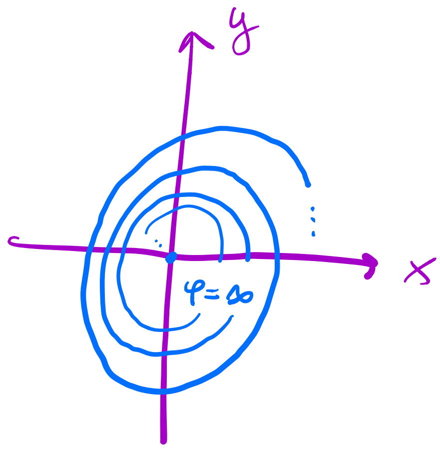




POZITIVNI DEL OS X SEKAMO PRI $a + 2k\pi b$.

(d) $r(\varphi) = \frac{a}{\varphi}, \quad \varphi \in (0, \infty)$



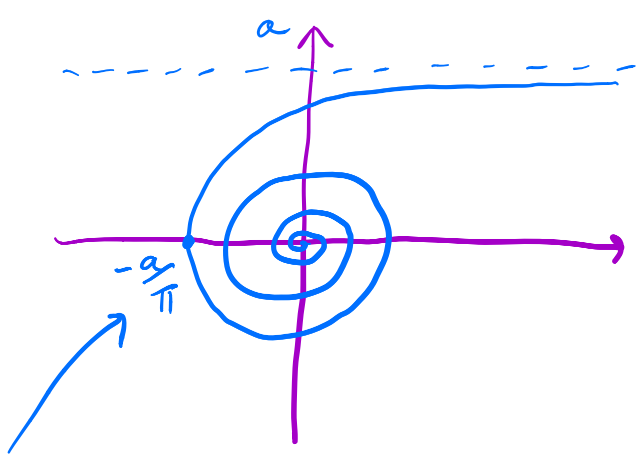


KAJ SE ZARES DOGAJA, KO $\varphi \downarrow 0$?

$$x(\varphi) = r(\varphi)\cos\varphi = \frac{a}{\varphi}\cos\varphi \longrightarrow \infty$$

$$y(\varphi) = r(\varphi)\sin\varphi = \frac{a}{\varphi}\sin\varphi \longrightarrow a$$

BOLJŠA SLIKA:



PRVO PRESEČIŠČE Z OSJO X: $y = \frac{a}{\varphi}\sin\varphi = 0 \Rightarrow \varphi = \pi$

$$x(\pi) = \frac{a}{\pi}\cos\pi = -\frac{a}{\pi}$$

DODATEK: KAKO IZRAČUNAMO TANGENTO NA TAKO KRIVULJO?

$$\boxed{k = \frac{y'(\varphi)}{x'(\varphi)}}$$

ENAKA FORMULA KOT PRI PARAM. KRIVULJAH

NPR. PRIMER (a) v TOČKI $\varphi = \frac{\pi}{4}$.

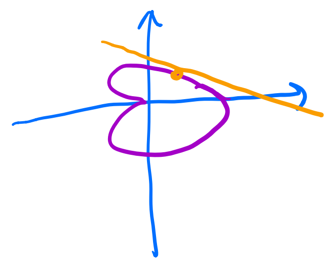
$$x(\varphi) = a(1+\cos\varphi)\cos\varphi$$
$$y(\varphi) = a(1+\cos\varphi)\sin\varphi$$
$$x'(\varphi) = -a\sin\varphi\cos\varphi - a(1+\cos\varphi)\sin\varphi$$
$$y'(\varphi) = -a\sin^2\varphi + a(1+\cos\varphi)\cos\varphi$$
$$x'(\tfrac{\pi}{4}) = -a\cdot\frac{1}{2} - a\left(1+\frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2} = $$
$$= -a\left(1+\frac{\sqrt{2}}{2}\right)$$
$$y'(\tfrac{\pi}{4}) = -a\cdot\frac{1}{2} + a\left(1+\frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2} = $$
$$= a\cdot\frac{\sqrt{2}}{2}$$
$$k = \frac{a\frac{\sqrt{2}}{2}}{-a\left(1+\frac{\sqrt{2}}{2}\right)} = -\frac{\sqrt{2}}{2+\sqrt{2}}$$

ENAČBA: $x(\tfrac{\pi}{4}) = a\left(1+\frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2}$

$y(\tfrac{\pi}{4}) = a\left(1+\frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2}$

$$\boxed{y - y(\tfrac{\pi}{4}) = k\left(x - x(\tfrac{\pi}{4})\right)}$$

PORAČUNAJ SAM!



NALOGA: S POMOČJO POLARNIH KOORDINAT SKICIRAJ LEMNISKATO, KI JE PODANA S PREDPISOM $(x^2+y^2)^2 = a^2(x^2-y^2)$, $a>0$.

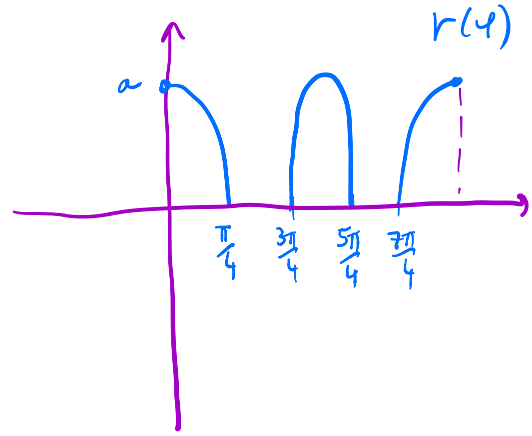
REŠITEV: VSTAVIMO $x = r\cos\varphi$ IN $y = r\sin\varphi$:

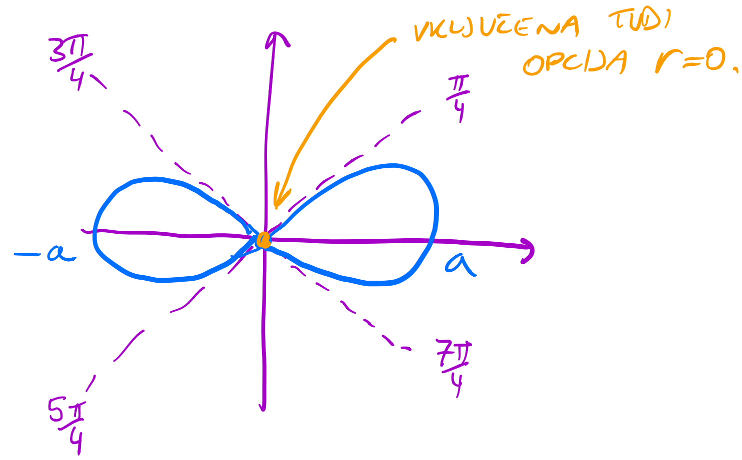
$$r^4 = a^2(r^2\cos^2\varphi - r^2\sin^2\varphi) \quad /: r^2$$

PREDPOSTAVIMO, DA $r \neq 0$.

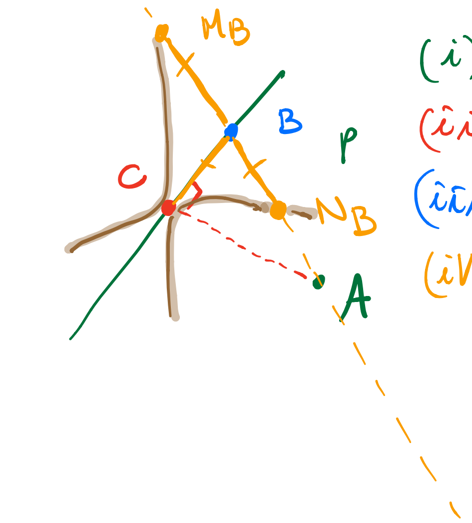
$$r^2 = a^2\cos 2\varphi$$
$$r = a\sqrt{\cos 2\varphi}$$

TA ZVEZA IMA SMISEL ZA $\varphi \in [0, \frac{\pi}{4}] \cup [\frac{3\pi}{4}, \frac{5\pi}{4}] \cup [\frac{7\pi}{4}, 2\pi]$.





NALOGA: SKICIRAJ PO KORAKIH:



(i) PREMICO p IN TOČKO $A \notin p$

(ii) C PRAVOKOTNA PROJEKCIJA A NA p

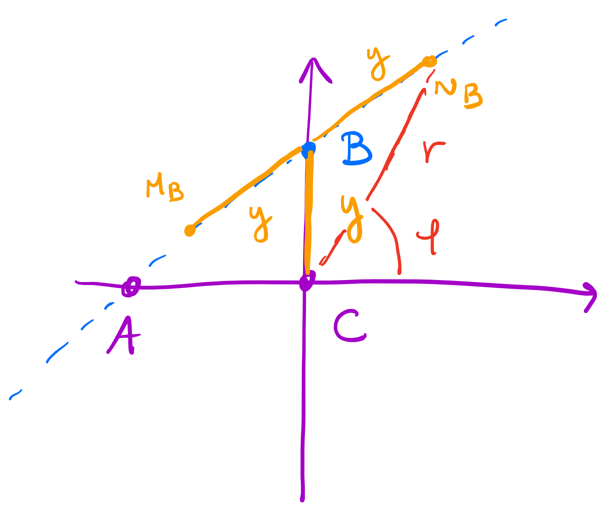
(iii) $B \in p$

(iv) TOČKI M_B IN N_B NA PREMICI SKOZI A IN B, KI ZADOŠČATA $|N_B B| = |M_B B| = |BC|$.

(v) KO PREMIKAŠ B, SE PREMIKATA TUDI N_B IN M_B. DOBIŠ KRIVULJO, KI JI PRAVIMO STROFOIDA

POIŠČI IZRAŽAVO STROFOIDE V POLARNIH KOORDINATAH IN JO SKICIRAJ.

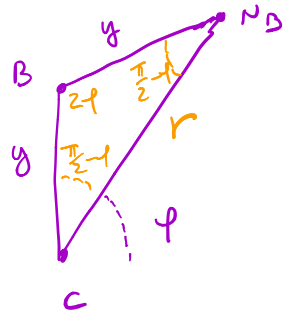
REŠITEV: BREZ ŠKODE ZA SPLOŠNOST $C(0,0)$ in $A(-1,0)$. PREMICA p PA USTREZA OSI y. TOČKA B ZAVZAME KOORDINATE $B(0,y)$, $y \in \mathbb{R}$.



ZA TA y STA VZDOLŽ AB OD B ODDALJENI TUDI M_B IN N_B.

OSREDOTOČIMO SE NA N_B IN PREDPOSTAVIMO, DA LEŽI V PRVEM KVADRANTU. OZNAČIMO NJEN POLARNI KOT S $\varphi \in [0, \frac{\pi}{2}]$ IN POLARNI RADIJ Z r.

TRIKOTNIK ΔCBN_B JE ENAKOKRAK, ZATO JE KOT $\sphericalangle CBN_B = 2\varphi$.



KOSINUSNI IZREK POVE:

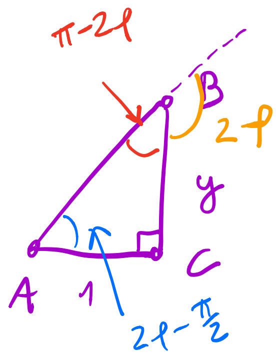
$$r^2 = y^2 + y^2 - 2y\cdot y\cos(2\varphi) = 2y^2(1-\cos 2\varphi)$$

$$= 2y^2(1-\cos^2\varphi + \sin^2\varphi) = 4y^2\sin^2\varphi$$

t.j. $r = +2y\sin\varphi$

($y>0$, V PRVEM KVADRANTU PA JE TUDI $\sin\varphi \geq 0$).

SEDAJ IZRAZIMO y S POMOČJO φ PREKO TRIKOTNIKA ΔABC.



$$\frac{y}{1} = \frac{\text{NASPROTNA}}{\text{PRILEŽNA}} = \operatorname{tg}\left(2\varphi - \frac{\pi}{2}\right)$$

$$y = \operatorname{tg}\left(2\varphi - \frac{\pi}{2}\right)$$

ZDRUŽIMO:

$$r = 2y\sin(\varphi) = 2\operatorname{tg}\left(2\varphi-\frac{\pi}{2}\right)\sin(\varphi) =$$

$$= -2\operatorname{ctg}(2\varphi)\sin(\varphi) = -2\frac{\cos(2\varphi)\sin\varphi}{\sin(2\varphi)} =$$

$$= -2\frac{\cos(2\varphi)\sin\varphi}{2\sin\varphi\cos\varphi} = -\frac{\cos(2\varphi)}{\cos(\varphi)}$$

TA PREDPIS JE POZITIVEN IN DOBRO DEFINIRAN LE ZA $\varphi \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right)$.

POSKUSI DOKAZATI SAM:

~ ENAK PREDPIS DOBIMO ZA N_B V 4. KVADRANTU, LE DA DRŽI ZA $\varphi \in \left(\frac{3\pi}{2}, \frac{7\pi}{4}\right]$

~ ENAK PREDPIS DOBIMO ZA M_B ZA $\varphi \in \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$.

REŠITEV BO OBJAVLJENA NA UČILNICI !!!

SKICIRAJMO SEDAJ KRIVULJO $r(\varphi) = -\dfrac{\cos(2\varphi)}{\cos(\varphi)}$.

OBMOČJE POZITIVNOSTI: $\left[\frac{\pi}{4}, \frac{\pi}{2}\right) \cup \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right] \cup \left(\frac{3\pi}{2}, \frac{7\pi}{4}\right]$

NIČLE: $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

POLI: $\frac{\pi}{2}, \frac{3\pi}{2}$.

STAC. TOČKE:

$$r' = + \frac{2\sin(2\varphi)\cos\varphi - \cos(2\varphi)\sin\varphi}{\cos^2\varphi} =$$

$$= \sin\varphi \, \frac{\sin^2\varphi + 3\cos^2\varphi}{\cos^2\varphi} = 0 \Rightarrow \boxed{\varphi = \pi}$$

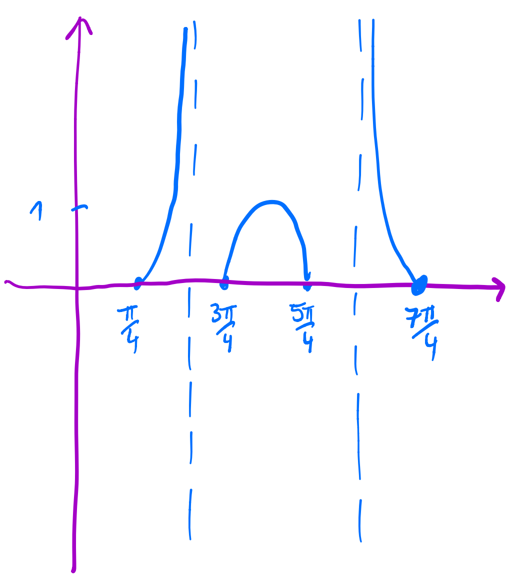
$$r(\pi) = -\frac{\cos(2\pi)}{\cos(\pi)} = 1.$$

KAJ SE ZGODI PRI $\varphi \to \frac{\pi}{2}$ IN $\varphi \to \frac{3\pi}{2}$:

$$x(\varphi) = -\frac{\cos(2\varphi)}{\cancel{\cos(\varphi)}} \cdot \cancel{\cos}\,\varphi \to 1$$

$$y(\varphi) = -\frac{\cos(2\varphi)}{\cos(\varphi)} \sin\varphi \to \pm\infty$$

$r(\varphi)$



STROFOIDA