MATEMATIKA 1

2. kolokvij (1.3.2021)

REŠITVE

1. naloga (25 točk)

Določi vsa stekališča zaporedja

$$a_n = e^{\sin\frac{n\pi}{2}} - \left(\frac{n^2+1}{n^2+2}\right)^{(-1)^n n^2+2n}.$$

$$e^{\sin\frac{n\pi}{2}} = \begin{cases} e^0 = 1 ; & n = 2k \\ e ; & n = 4k+1 \\ e^{-1} ; & n = 4k+3 \end{cases} \quad 5$$

$$\frac{(-1)^n n^2 + 2n}{-n^2-2} \to (-1)^{n+1}$$

$$\left(\frac{n^2+1}{n^2+2}\right)^{(-1)^n n^2+2n} = \underbrace{\left(\left(1-\frac{1}{n^2+2}\right)^{-(n^2+2)}\right)}_{\downarrow\ e} \longrightarrow \underbrace{\begin{cases} e ; & n \text{ lih} \\ e^{-1} ; & n \text{ sodo} \end{cases}}_{15}$$

$$\Longrightarrow s_1 \overset{n \text{ sodo}}{=} 1 - e^{-1},\quad s_2 \overset{n=4k+1}{=} 0,\quad s_3 \overset{n=4k+3}{=} e^{-1} - e \Big\} \quad 5$$

2. naloga

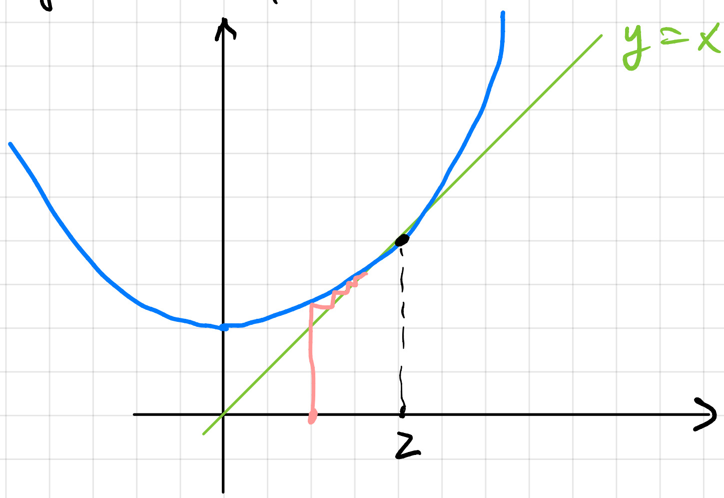
Zaporedje je podano z začetnim členom $a_1 = a > 0$ in rekurzivno zvezo

$$a_{n+1} = 1 + \frac{a_n^2}{4}.$$

a) (18 točk) Ugotovi ali zaporedje konvergira za $a = 1$ in v primeru konvergence določi še limito $\lim\limits_{n\to\infty} a_n$. Vse sklepe natančno utemelji.

b) (7 točk) Grafično preveri konvergenco za poljuben $a > 0$.

a) $f(x) = 1 + \frac{x^2}{4} = x \Longrightarrow x^2 - 4x + 4 = 0 \Longrightarrow \underline{\underline{x = 2}}$



5

Zaporedje raste in $a_n \le 2$.

1. $a_n \le 2$:

$\boxed{n=1}$ $\quad a_1 = 1 \le 2$ ✓ (očitno $a_n \ge 0$)

$\boxed{\text{I.K.}}$ $\quad a_{n+1} = 1 + \dfrac{a_n^2}{4} \le 1 + \dfrac{4}{4} = 2$ ✓ 5

$a_n^2 \le 4$

2. $a_{n+1} - a_n = 1 + \dfrac{a_n^2}{4} - a_n = \dfrac{a_n^2 - 4a_n + 4}{4} =$

$= \dfrac{(a_n - 2)^2}{4} \ge 0 \Rightarrow a_n$ raste ✓ 5

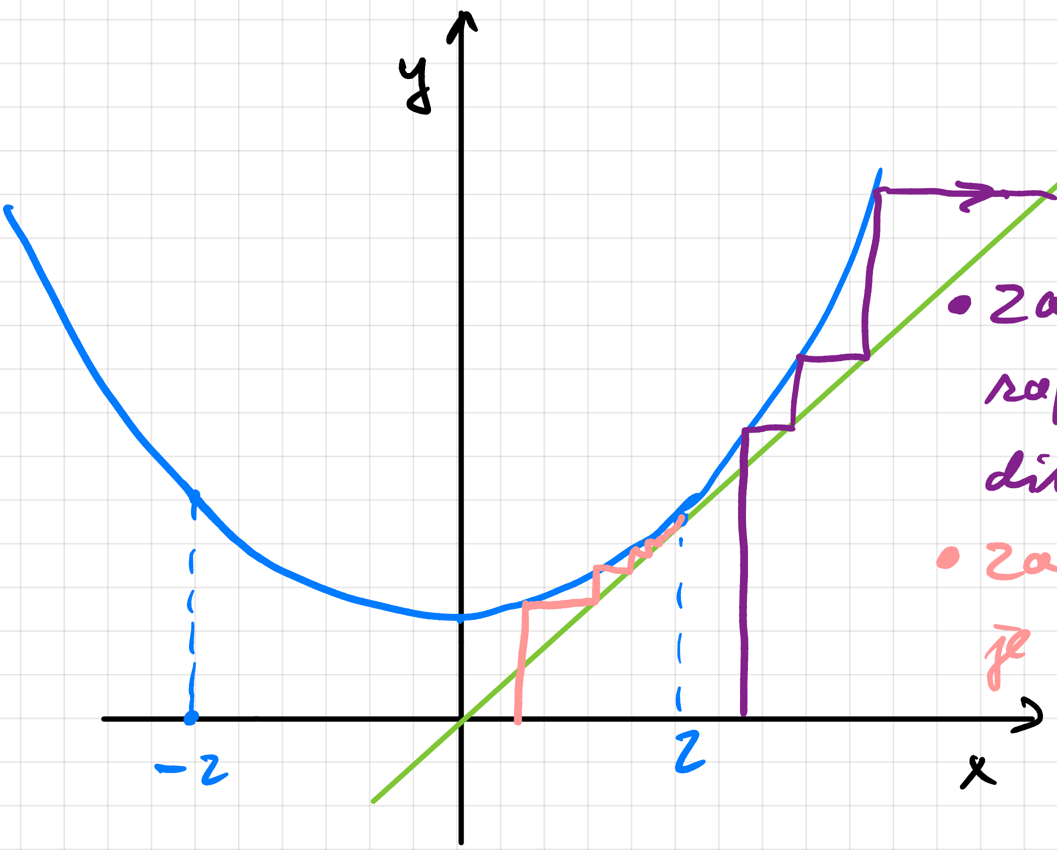
3. ① & ② $\Rightarrow \exists \lim\limits_{n\to\infty} a_n = \alpha$

$$1 + \frac{a_n^2}{4} = a_{n+1}$$

$$1 + \frac{\alpha^2}{4} = \alpha \Rightarrow \underline{\underline{\alpha = 2}}$$

3

b)



• za $a > 2$ zaporedje divergira 4

• za $a \in [0,2]$ je $\lim\limits_{n\to\infty} a_n = 2$ 3

3. naloga

a) (10 točk) Ugotovi ali konvergira vrsta

$$\sum_{n=1}^{\infty} \underbrace{\frac{(2n)!}{(n!)^2 8^n}}_{a_n}.$$

b) (15 točk) Obravnavaj konvergenco in absolutno konvergenco vrste

$$\sum_{n=1}^{\infty} \underbrace{\frac{\sqrt{n+1}-\sqrt{n}}{(-1)^n n^s}}_{a_n}$$

glede na parameter $s > 0$.

a) $$D_n = \frac{a_{n+1}}{a_n} = \frac{(2n+2)!\,\cancel{(n!)^2}\,\cancel{8^n}}{((n+1)!)^2\, 8^{\cancel{n+1}} \cdot (2n)!} = \frac{(2n+2)(2n+1)}{(n+1)^2 \cdot 8}$$ 5

$$\lim_{n\to\infty} D_n = \frac{4}{8} = \frac{1}{2} < 1 \implies \text{vrsta konvergira}$$ 5

b) $$a_n = \frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{(-1)^n n^s (\sqrt{n+1}+\sqrt{n})} = \frac{1}{(-1)^n n^s (\sqrt{n+1}+\sqrt{n})}$$ 5

ABSOLUTNA KONV.:

$$|a_n| = \frac{1}{n^s(\sqrt{n+1}+\sqrt{n})} \approx \frac{1}{2n^{s+1/2}}$$

L.P.K. $$\lim_{n\to\infty} \frac{|a_n|}{\frac{1}{n^{s+1/2}}} = \lim_{n\to\infty} \frac{\frac{1}{\cancel{n^{s+\frac{1}{2}}}(\sqrt{1+1/n}+1)}}{\frac{1}{\cancel{n^{s+1/2}}}} = \frac{1}{2} \neq 0$$

$\implies$ vrsta konv. $\iff$ vrsta $\sum_1^{\infty} \frac{1}{n^{s+1/2}}$ konv. $\iff s+1/2 > 1 \iff s > 1/2$ 5

KONV. Zaporedje $c_n = \frac{1}{n^s(\sqrt{n+1}+\sqrt{n})}$ pada za $s > 0 \xRightarrow{\text{Leib.}}$ konvergira. 5

4. naloga

Dana je funkcija

$$f(x) = \frac{4x^2 - 36}{x^2 - 2x - 8}.$$

a) (15 točk) Skiciraj graf funkcije f.

b) (10 točk) Naj bo $g(x) = \operatorname{arctg}(\ln f(x))$. Določi definicijsko območje funkcije g in skiciraj njen graf.

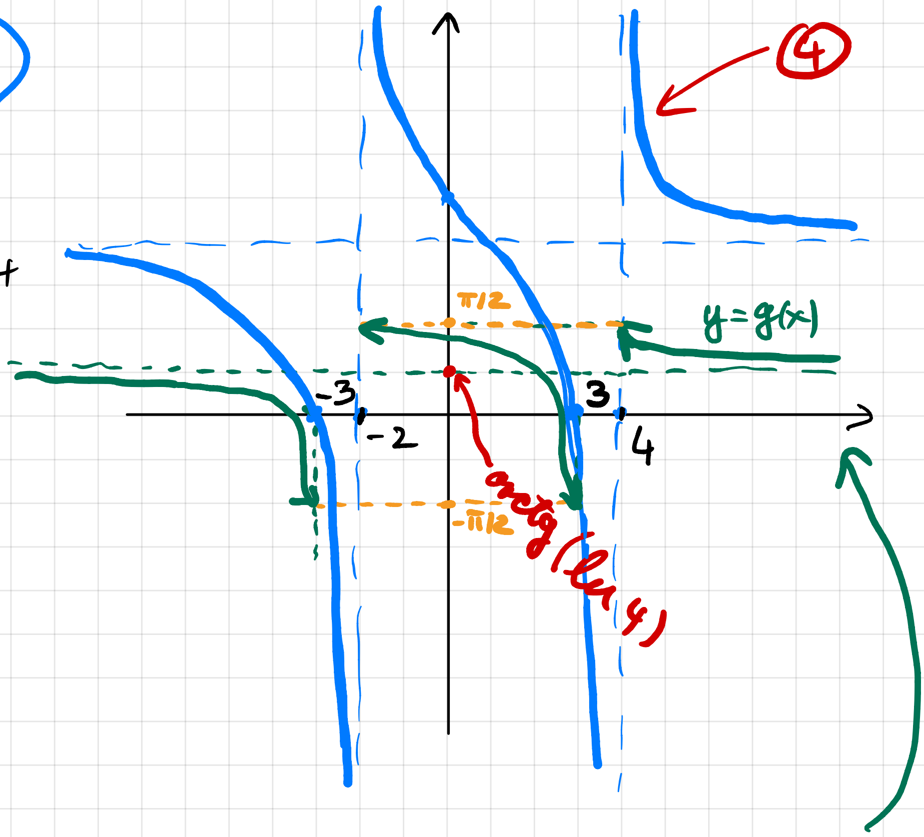
a) $\boxed{f(x) = \frac{4(x-3)(x+3)}{(x-4)(x+2)}}$

3 Ničli: $x = \pm 3$

3 Pola: $x = -2$, $x = 4$

3 $\lim_{x \to \pm\infty} f(x) = 4$

2 $f(0) = \frac{9}{2}$



b) $\ln f(x)$ obstaja $\iff f(x) > 0 \iff D_g = (-\infty, -3) \cup (-2, 3) \cup (4, \infty)$

5,

graf = 5