FOURIERJEVA VRSTA

Naj bo f funkcija, ki je integrabilna na $(-\pi,\pi)$.

TRIGONOMETRIJSKA (KLASIČNA) FOURIERJEVA VRSTA:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\sin(nx)\,dx$$

$$\tilde{f}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cdot\cos(nx) + \sum_{n=1}^{\infty} b_n\cdot\sin(nx)$$

Lastnosti:

- $\tilde{f}$ je periodična s periodo 2π
- ZA ZVEZNE f: $\tilde{f}(\pi) = \tilde{f}(-\pi) = \frac{1}{2}\left(\lim_{x\nearrow\pi} f(x) + \lim_{x\downarrow-\pi} f(x)\right)$

  na $(-\pi,\pi)$ se f in $\tilde{f}$ ujemata
- v splošnem: na $(-\pi,\pi)$ se f in $\tilde{f}$ ujemata, razen v točkah nezveznosti (tam $\tilde{f}$ zavzame aritmetično sredino limit).
- PARSEVALOVA ENAČBA: $\int_{-\pi}^{\pi} |f(x)|^2dx = \frac{\pi}{2}a_0^2 + \pi\cdot\sum_{n=1}^{\infty}(a_n^2+b_n^2)$

(10): $f(x) = \sin^4 x$

a) Funkcijo f razvij v trigonometrijsko Fourierjevo vrsto na $[-\pi,\pi]$

b) Za vse $n \in \mathbb{N} \cup \{0\}$ izračunaj $\int_{-\pi}^{\pi} \sin^4 x \cos kx \, dx$.

c) Izračunaj $\int_{-\pi}^{\pi} \sin^8 x \, dx$

a) $$\sin^4 x = (\sin^2 x)^2 = \left(\tfrac{1}{2}(1-\cos 2x)\right)^2 = \tfrac{1}{4}(1 - 2\cos 2x + \cos^2 2x) =$$

$\sin^2 x + \cos^2 x = 1$

$\cos^2 x - \sin^2 x = \cos 2x$

$1 - \sin^2 x - \sin^2 x = \cos 2x \Rightarrow \sin^2 x = \frac{1-\cos 2x}{2}$

$\cos^2 2x + \sin^2 2x = 1$

$\cos^2 2x - \sin^2 2x = \cos 4x$

$\cos^2 2x = \frac{1+\cos 4x}{2}$

$$= \tfrac{1}{4} - \tfrac{1}{2}\cos 2x + \tfrac{1}{4}\left[\tfrac{1}{2}(1+\cos 4x)\right] = \tfrac{1}{4} - \tfrac{1}{2}\cos 2x + \tfrac{1}{8} + \tfrac{1}{8}\cos 4x =$$

$$= \underline{\underline{\tfrac{3}{8} - \tfrac{1}{2}\cos 2x + \tfrac{1}{8}\cos 4x}}$$

b) $$f(x) = \tfrac{1}{2}a_0 + \sum_{k=1}^{\infty} a_n \cdot \cos(nx) + b_n \cdot \sin(nx) \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos(nx)\,dx$$

• $\underline{\underline{n=0}}$: $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$ od zgoraj: $\frac{1}{2}a_0 = \frac{3}{8} \Rightarrow a_0 = \frac{3}{4}$

$$\Rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^4 x \, dx = \frac{3}{4} \Rightarrow \underline{\underline{\int_{-\pi}^{\pi} \sin^4 x = \frac{3\pi}{4}}}$$

• $\underline{\underline{n=1}}$: $a_1 = 0 \Rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^4 x \cdot \cos x \, dx = 0 \Rightarrow \underline{\underline{\int_{-\pi}^{\pi} \sin^4 x \cdot \cos x \, dx = 0}}$

• $\underline{\underline{n=2}}$: $a_2 = -\frac{1}{2}$ (iz a) primera)

$$a_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^4 x \cdot \cos 2x \, dx \Rightarrow \underline{\underline{\int_{-\pi}^{\pi} \sin^4 x \cdot \cos 2x \, dx = -\frac{\pi}{2}}}$$

• $\underline{\underline{n=3}}$: podobno kot $n=1$

• $\underline{\underline{n=4}}$: $a_4 = \frac{1}{8} \Rightarrow \underline{\underline{\int_{-\pi}^{\pi} \sin^4 x \cos 4x = \frac{\pi}{8}}}$

• za vse ostale n: $\int_{-\pi}^{\pi} \ldots = 0$

c) Parsevalova enakost:

10.3.2021

$$\boxed{\int_{-\pi}^{\pi} (f(x))^2\,dx = \frac{\pi}{2}a_0^2 + \pi\sum_{n=1}^{\infty} a_n^2+b_n^2}$$

$$\int_{-\pi}^{\pi} \sin^8 x\,dx = \int_{-\pi}^{\pi} (\sin^4 x)^2\,dx = \frac{\pi}{2}\cdot\left(\frac{3}{4}\right)^2 + \pi\cdot\left(\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{8}\right)^2\right) =$$

$$= \underline{\underline{\frac{\pi}{2^5}\cdot 9 + \pi\cdot\left(\frac{1}{4}+\frac{1}{8^2}\right)}} = \underline{\underline{\frac{75\pi}{128}}}$$

N: $f(x) = \sin(x+\frac{\pi}{6})$

a) Funkcijo f razvij v Fourierjevo na $[-\pi,\pi]$

b) Izračunaj $\int_{-\pi}^{\pi} f(x)\cdot\cos(nx)\,dx$ in $\int_{-\pi}^{\pi} f(x)\cdot\sin(nx)\,dx$

za vse $n\in\mathbb{N}$

a) Adicijski izreki:

$$\sin\left(x+\frac{\pi}{6}\right) = \sin x\cdot\cos\frac{\pi}{6} + \cos x\cdot\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2}\cdot\sin x + \frac{1}{2}\cdot\cos x$$

$$\Rightarrow \quad a_1 = \frac{1}{2} \qquad b_1 = \frac{\sqrt{3}}{2}$$

b)

$$\int_{-\pi}^{\pi} f(x)\cdot\cos(nx)\,dx = \pi\cdot a_n = \begin{cases} \frac{\pi}{2}; & n=1 \\ 0; & n\neq 1 \end{cases}$$

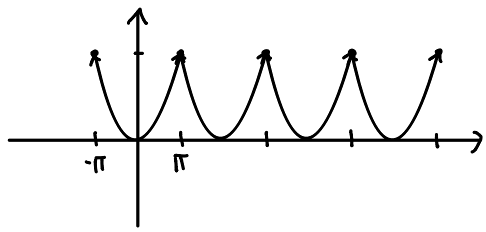
$$\int_{-\pi}^{\pi} f(x)\cdot\sin nx\,dx = \pi\cdot b_n = \begin{cases} \frac{\sqrt{3}\pi}{2}; & n=1 \\ 0; & n\neq 1 \end{cases}$$

(1): $f(x) = x^2$

a) Funkcijo f razvij v Fourierjevo vrsto na $[-\pi, \pi]$

b) Izračunaj $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

c) Izračunaj $\sum_{n=1}^{\infty} \frac{1}{n^4}$.



a) $\tilde{f}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos nx + b_n \cdot \sin nx$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2\,dx = \frac{1}{\pi} \left. \frac{x^3}{3} \right|_{-\pi}^{\pi} = \frac{2}{3}\pi^2$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cdot \cos nx\,dx \underset{\text{per partes}}{=} \frac{1}{\pi} \cdot \left(x^2 \left. \frac{\sin nx}{n} \right|_{-\pi}^{\pi} - \frac{2}{n} \int_{-\pi}^{\pi} x \cdot \sin nx\,dx \right) =$$

$u = x^2 \qquad du = 2x\,dx$

$dv = \cos nx\,dx \qquad v = \frac{\sin nx}{n}$

$$= \underbrace{\frac{1}{\pi} \cdot x^2 \cdot \frac{1}{n} \cdot \sin nx \Big|_{-\pi}^{\pi}}_{0} - \frac{2}{\pi n} \int_{-\pi}^{\pi} x \cdot \sin nx\,dx = -\frac{2}{\pi n} \cdot \left(x \cdot \left(-\frac{\cos nx}{n} \right) \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{\cos nx}{n}\,dx \right) =$$

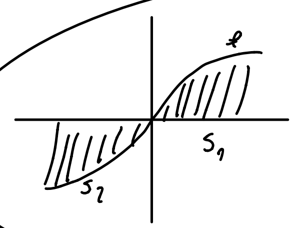
$u = x \qquad du = dx$

$dv = \sin nx\,dx \qquad v = \frac{\cos nx}{n}$

$$= \frac{2}{\pi n^2} \cdot x \cdot \cos nx \Big|_{-\pi}^{\pi} + \underbrace{\frac{2}{\pi n^2} \cdot \frac{\sin nx}{n} \Big|_{-\pi}^{\pi}}_{0} = \frac{2}{\pi n^2} \cdot \left(\pi \cdot \cos(n\pi) - (-\pi) \cdot \cos(n(-\pi)) \right) =$$

$$= \frac{2}{\pi n^2} \cdot 2\pi \cdot (-1)^n = \underline{\underline{\frac{4(-1)^n}{n^2}}}$$

$$b_n = \underbrace{\frac{1}{\pi} \int_{-\pi}^{\pi} \overset{\text{soda}}{x^2} \cdot \overset{\text{liha}}{\sin nx}\,dx}_{\text{liha}} = 0$$



$\int_{-\pi}^{\pi} f\,dx = S_1 - S_2 = 0$

$$\underline{\underline{\tilde{f}(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4 \cdot (-1)^n}{n^2} \cdot \cos(nx)}}$$

b) moti nas $(-1)^n \cdot \cos(nx)$

brez tega bi imeli: $\frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} = \frac{\pi^2}{3} + 4 \cdot \sum_{n=1}^{\infty} \frac{1}{n^2}$

$(-1)^n \cdot \cos(nx) = 1$, če je $\cos(nx) = (-1)^n$ $\leftarrow$ to dosežemo pri $x = \pi$:

$$\tilde{f}(\pi) = \frac{\pi^2}{3} + 4 \cdot \sum_{n=1}^{\infty} \frac{1}{n^2}$$

po drugi strani pa vemo: $\tilde{f}(\pi) = \frac{1}{2}\left(\lim_{x \nearrow \pi} f(x) + \lim_{x \searrow -\pi} f(x)\right) =$

$$= \frac{1}{2}(\pi^2 + \pi^2) = \pi^2$$

$$\Rightarrow \quad \pi^2 = \frac{\pi^2}{3} + 4 \cdot \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\pi^2\left(1 - \frac{1}{3}\right) = 4 \cdot \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{4} \cdot \frac{2}{3}$$

c) Parsevalova enakost:

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{\pi}{2} a_0^2 + \pi \sum_{n=1}^{\infty} a_n^2 + \underbrace{b_n^2}_{0}$$

$$\int_{-\pi}^{\pi} x^4 dx = \frac{\pi}{2} \cdot \left(\frac{2\pi^2}{3}\right)^2 + \pi \cdot \sum_{n=1}^{\infty} \left(\frac{4 \cdot (-1)^n}{n^2}\right)^2$$

$$\left.\frac{x^5}{5}\right|_{-\pi}^{\pi} = \frac{4\pi^5}{18} + \pi \cdot 16 \cdot \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\frac{\pi^5 + \pi^5}{5} = \frac{2\pi^5}{9} + 16\pi \cdot \sum_{n=1}^{\infty} \frac{1}{n^4}$$

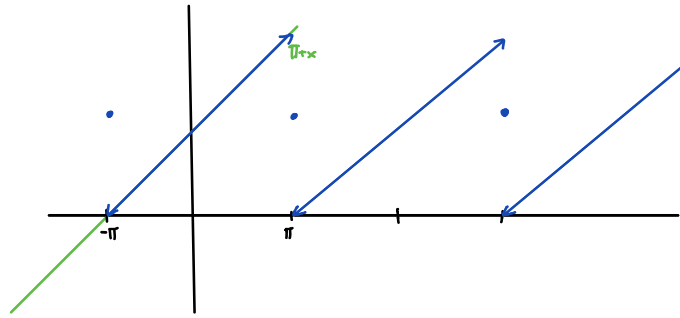
$$\frac{2\pi^5}{5} - \frac{2\pi^5}{9} = 16\pi \cdot \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$2\pi^4 \frac{9-5}{45} = 16 \cdot \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

5: $f(x) = \pi + x$

Funkcijo f razvij v Fourierjevo vrsto na $[-\pi, \pi]$ ter nariši funkciji f in $\tilde{f}$



$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx =$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi + x)\,dx =$$

$$= \frac{1}{\pi} \cdot \left(\pi x + \frac{x^2}{2}\right)\Big|_{-\pi}^{\pi} =$$

$$= \frac{1}{\pi} \left(\pi^2 - (-\pi^2) + \frac{\pi^{\cancel{2}}}{\cancel{2}} - \frac{\pi^{\cancel{2}}}{\cancel{2}}\right) =$$

$$= \frac{1}{\cancel{\pi}} \cdot 2\pi^{\cancel{2}} = 2\pi$$

$$a_n = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} (\pi + x) \cdot \cos(nx)\,dx = \frac{1}{\pi} \cdot \left((\pi + x) \cdot \frac{\sin nx}{n}\right)\Big|_{-\pi}^{\pi} - \frac{1}{\pi n} \int_{-\pi}^{\pi} \sin nx\,dx =$$

per partes

$u = (\pi + x)$ $\quad du = dx$

$dv = \cos(nx)$ $\quad v = \frac{\sin nx}{n}$

$$= 0 - \frac{1}{\pi \cdot n} \cdot \left(-\frac{\cos(nx)}{n}\right)\Big|_{-\pi}^{\pi} = 0$$

$\sin(n\pi) = 0$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + \pi) \cdot \sin(nx)\,dx = \frac{1}{\pi} \cdot \left((\pi + x) \cdot \left(-\frac{\cos nx}{n}\right)\Big|_{-\pi}^{\pi}\right) + \frac{1}{\pi n} \cdot \int_{-\pi}^{\pi} \cos nx\,dx =$$

$$= -\frac{1}{\pi} \cdot \left((\pi + \pi) \cdot \left(-\frac{\cos(n\pi)}{n}\right) - (\pi - \pi) \cdot \left(-\frac{\cos(-n\pi)}{n}\right)\right) + \frac{1}{n\pi} \int_{-\pi}^{\pi} \cos nx\,dx =$$

$$= + \frac{2\cancel{\pi}}{\cancel{\pi}} \cdot \frac{\cos(n\pi)}{n} - 0 + \frac{1}{n\pi} \cdot \frac{\sin nx}{n}\Big|_{-\pi}^{\pi} =$$

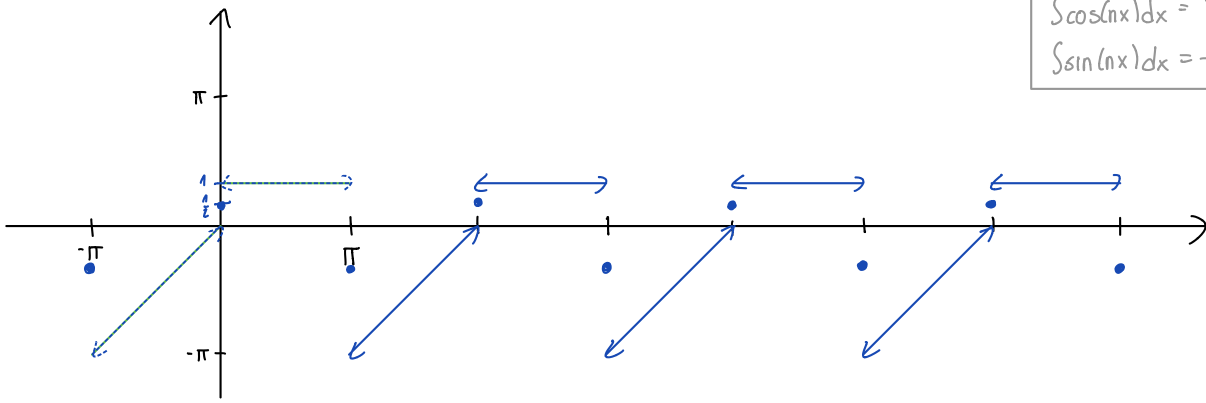
$$= \frac{2 \cdot (-1)^{n+1}}{n} + 0 = \frac{2(-1)^{n+1}}{n}$$

$$\tilde{f}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos(nx) + b_n \cdot \sin(nx) = \frac{2\pi}{2} + \sum_{n=1}^{\infty} 0 \cdot \cos(nx) + \frac{2(-1)^{n+1}}{n} \cdot \sin(nx) =$$

$$= \pi + 2 \cdot \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot \sin(nx)$$

(1.) Funkcijo $f(x)=\begin{cases} x \ ; -\pi < x \le 0, \\ 1 \ ; \ 0 < x \le \pi. \end{cases}$ razvij v trigonometrijsko Fourierjevo vrsto.

$$\int \cos(nx)\,dx = \frac{\sin(nx)}{n} + C$$
$$\int \sin(nx)\,dx = -\frac{\cos(nx)}{2} + C$$



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} x\,dx + \int_0^{\pi} 1\,dx\right) = \frac{1}{\pi}\left(\frac{x^2}{2}\Big|_{-\pi}^{0} + \pi\right) = \frac{1}{\pi}\left(-\frac{\pi^2}{2}+\pi\right) = 1-\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\left(\int_{-\pi}^{0} x\cdot\cos nx\,dx + \int_0^{\pi}\cos nx\,dx\right) = \frac{1}{\pi}\cdot\left(x\cdot\frac{\sin nx}{n}\Big|_{-\pi}^{0} - \int_{-\pi}^{0}\frac{\sin nx}{n}\,dx + \frac{\sin nx}{n}\Big|_0^{\pi}\right) =$$

$u = x \quad du = dx$

$dv = \cos nx\,dx \quad v = \frac{\sin nx}{n}$

$$= \frac{1}{\pi}\cdot\left(0 - \frac{1}{n}\cdot\left(-\frac{\cos nx}{n}\right)\Big|_{-\pi}^{0} + 0\right) = \frac{1}{\pi n^2}\cdot\left(\cos 0 - \cos(-n\pi)\right) = \frac{1}{\pi n^2}\left(1-\cos(n\pi)\right) =$$

$$= \underline{\underline{\frac{1}{\pi n^2}\cdot\left(1-(-1)^n\right)}}$$

$$b_n = \frac{1}{\pi}\left(\int_{-\pi}^{0} x\sin nx\,dx + \int_0^{\pi}\sin nx\,dx\right) = \frac{1}{\pi}\left(-x\cdot\frac{\cos nx}{n}\Big|_{-\pi}^{0} - \int_{-\pi}^{0}\left(-\frac{\cos nx}{n}\right)dx - \frac{\cos nx}{n}\Big|_0^{\pi}\right) =$$

$u = x \quad du = dx$

$dv = \sin nx\,dx \quad v = -\frac{\cos nx}{n}$

$$= \frac{1}{\pi}\left(0 - \frac{\pi}{n}\cdot\cos(-n\pi) + \underbrace{\frac{1}{n}\cdot\frac{\sin nx}{n}\Big|_{-\pi}^{0}}_{0} - \frac{\cos n\pi}{n} + \frac{\cos n0}{n}\right) =$$

$$= \frac{1}{\pi}\left(-\frac{\pi}{n}\cdot(-1)^n - \frac{1}{n}\cdot(-1)^n + \frac{1}{n}\right) = \frac{1}{\pi n}\cdot\left((-1)^n\cdot(-\pi-1)+1\right)$$

če n sod, je to 0

$$\tilde{f}(x) = \frac{1}{2}\cdot\left(1-\frac{\pi}{2}\right) + \sum_{n=1}^{\infty}\frac{1-(-1)^n}{\pi n^2}\cdot\cos(nx) + \sum_{n=1}^{\infty}\frac{1-(-1)^n(1+\pi)}{\pi n}\cdot\sin nx =$$

$$= \frac{1}{2}\cdot\left(1-\frac{\pi}{2}\right) + \sum_{k=1}^{\infty}\frac{2}{\pi\cdot(2k-1)^2}\cdot\cos(2k-1)x + \sum_{n=1}^{\infty}\frac{1-(-1)^n\cdot(1+\pi)}{\pi n}\cdot\sin nx$$

Trigonometrijska Fourierjeva vrsta na poljubnem intervalu

Naj bo f integrabilna na (u,v)

$$a_0 = \frac{2}{v-u}\int_u^v f(x)\,dx$$

$$a_n = \frac{2}{v-u}\int_u^v f(x)\cdot\cos\left(n\cdot\frac{2\pi}{v-u}\cdot x\right)dx$$

$$b_n = \frac{2}{v-u}\int_u^v f(x)\cdot\sin\left(n\cdot\frac{2\pi}{v-u}\cdot x\right)dx$$

- enake lastnosti kot pri klasični F.V.:

  v točkah zveznosti na (u,v) se f in $\tilde{f}$ ujemata, sicer aritmetična sredina limit

  $$\tilde{f}(x) = \frac{1}{2}\left(\lim_{y\uparrow x} f(y) + \lim_{y\downarrow x} f(y)\right)$$

  na robo:

  $$\tilde{f}(u) = \tilde{f}(v) = \frac{1}{2}\left(\lim_{x\downarrow u} f(x) + \lim_{x\uparrow v} f(x)\right)$$

  PARSEVALOVA ENAČBA:

  $$\int_u^v (f(x))^2\,dx = \frac{v-u}{4}\cdot a_0^2 + \frac{v-u}{2}\cdot\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

11: $f(x) = \begin{cases} x & ; 0 \le x \le 1 \\ 2-x & ; 1 < x \le 2 \end{cases}$ razvij v Fourierjevo vrsto na $[0,2]$.

$$u=0, \quad v=2 \quad \rightarrow \quad v-u=2, \quad \frac{2}{v-u}=1, \quad \frac{2\pi n x}{v-u} = \pi n x$$

$$a_0 = 1\cdot\int_0^2 f(x)\,dx = \int_0^1 x\,dx + \int_1^2 2-x\,dx = \frac{x^2}{2}\Big|_0^1 + 2x\Big|_1^2 - \frac{x^2}{2}\Big|_1^2 = \frac{1}{2}-0+\cancel{4}-\cancel{2}-\cancel{\frac{4}{2}}+\frac{1}{2} = 1$$

$$a_n = 1\cdot\int_0^2 f(x)\cdot\cos(\pi n x)\,dx = \int_0^1 x\cdot\cos(\pi n x)\,dx + \int_1^2 (2-x)\cdot\cos(\pi n x)\,dx =$$

$u=x \quad dv=\cos(\pi n x)dx$

$du=dx \quad v=\frac{1}{\pi n}\sin(\pi n x)$

$$= \int_0^1 x\cos(\pi n x)\,dx - \int_1^2 x\cdot\cos(\pi n x)\,dx + \int_1^2 2\cos(\pi n x)\,dx \overset{P.P.}{=}$$

$$= \frac{x}{\pi n}\cdot\sin(\pi n x)\Big|_0^1 - \frac{1}{\pi n}\int_0^1 \sin(\pi n x)\,dx - \frac{x}{\pi n}\cdot\sin(\pi n x)\Big|_1^2 + \frac{1}{\pi n}\int_1^2 \sin(\pi n x) + 2\cdot\frac{1}{\pi n}\cdot\sin(\pi n x)\Big|_1^2 =$$

$$= 0-0-\frac{1}{\pi n}\cdot\left(-\frac{1}{\pi n}\cdot\cos(\pi n x)\right)\Big|_0^1 - 0+0+\frac{1}{\pi n}\cdot\left(-\frac{1}{\pi n}\cdot\cos(\pi n x)\right)\Big|_1^2 + 0-0 =$$

$$= \frac{1}{(\pi n)^2}\left(\cos(\pi n)-\cos 0\right) - \frac{1}{(\pi n)^2}\cdot\left(\underbrace{\cos(2\pi n)}_{=\cos 0}-\cos(\pi n)\right) =$$

$$= \frac{1}{(\pi n)^2}\cdot\left((-1)^n-1-1+(-1)^n\right) = \frac{2}{(\pi n)^2}\cdot\left((-1)^n-1\right)$$

$$b_n = 1\cdot\int_0^2 f(x)\cdot\sin(\pi n x)\,dx = \int_0^1 x\cdot\sin(\pi n x)\,dx + \int_1^2 (2-x)\cdot\sin(\pi n x)\,dx =$$

$t=2-x$

$dt=-dx$

$$= \int_0^1 x\sin(\pi n x)\,dx + \int_1^0 t\cdot\sin(\pi n\cdot(2-t))(-dt)$$

$$= \int_0^1 x\sin(\pi n x)\,dx - \int_1^0 t\cdot\sin(2\pi n - \pi n t)\,dt =$$

$$= \int_0^1 x\sin(\pi n x)\,dx + \int_0^1 t\sin(-\pi n t)\,dt =$$

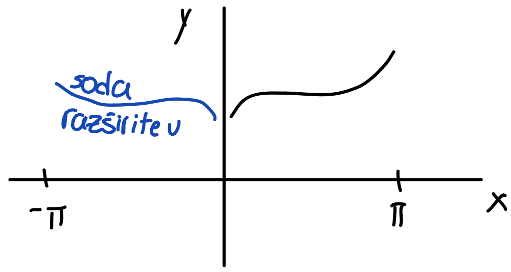
$$= \int_0^1 x\sin(\pi n x)\,dx - \int_0^1 t\sin(\pi n t)\,dt = 0$$

$$\tilde{f}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cdot\cos(\pi n x) + \sum_{n=1}^{\infty} b_n\cdot\sin(\pi n x) =$$

$$= \frac{1}{2} + \frac{2}{(\pi n)^2}\cdot\sum_{n=1}^{\infty}\left((-1)^n-1\right)\cdot\cos(\pi n x)$$

KOSINUSNA FOURIERJEVA VRSTA

Naj bo f integrabilna na $(0,\pi)$.



$$a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx, \quad a_n = \frac{2}{\pi}\int_0^\pi f(x)\cdot\cos(nx)\,dx$$

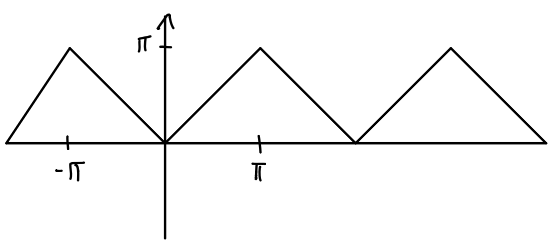
$$\tilde{f}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cdot\cos(nx)$$

Lastnosti:

- $\tilde{f}$ je soda
- $\tilde{f}$ je periodična s periodo 2π
- v točkah iz $(0,\pi)$, kjer je f zvezna: $\tilde{f} = f$
  kjer f ni zvezna: $\tilde{f}(x) = \frac{1}{2}\left(\lim_{y\nearrow x} f(y) + \lim_{y\searrow x} f(y)\right)$
- $\tilde{f}(0) = \lim_{y\searrow 0} f(y)$
  $\tilde{f}(\pi) = \lim_{y\nearrow \pi} f(y)$
- PARSEVALOVA ENAČBA: $\int_0^\pi (f(x))^2\,dx = \frac{\pi}{4}a_0^2 + \frac{\pi}{2}\sum_{n=1}^{\infty} a_n^2$

N: Razvij $f(x) = x$ v kosinusno Fourierjevo vrsto na intervalu $(0,\pi)$.

$$a_0 = \frac{2}{\pi}\int_0^\pi x\,dx = \frac{1}{\pi}x^2\Big|_0^\pi = \frac{\pi^2}{\pi} = \pi$$



$$a_n = \frac{2}{\pi}\int_0^\pi x\cdot\cos(nx)\,dx \overset{P.P.}{=}$$

$u = x \quad dv = \cos(nx)\,dx$
$du = dx \quad v = \sin(nx)\cdot\frac{1}{n}$

$$= \frac{2}{\pi}\left(\frac{x}{n}\sin(nx)\Big|_0^\pi - \frac{1}{n}\int_0^\pi \sin(nx)\,dx\right) = \frac{2}{\pi}\left(0 + \frac{1}{n^2}\cos(nx)\Big|_0^\pi\right) =$$

$$= \frac{2}{\pi n^2}\left(\cos(n\pi) - \cos 0\right) = \frac{2}{\pi n^2}\left((-1)^n - 1\right) = \begin{cases} -\frac{4}{\pi n^2}; & n \text{ lih} \\ 0; & n \text{ sod} \end{cases}$$

$$\tilde{f}(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\cdot\left((-1)^n - 1\right) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cdot\cos((2k-1)x)$$

11: Razvij $f(x)=\sin x$ v kosinusno Fourierjevo vrsto. *Dodatna naloga

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}\sin x\,dx = \frac{2}{\pi}\cdot(-\cos x)\Big|_0^{\pi} = \frac{2}{\pi}\big(-(-1)-(-1)\big) = \underline{\underline{\frac{4}{\pi}}}$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\cos x\,dx = \frac{2}{\pi}\int_0^{\pi}\sin x\cos x\,dx = \frac{1}{\pi}\int_0^{\pi}\sin(2x)\,dx = \frac{1}{\pi}\left(-\cos(2x)\cdot\frac{1}{2}\right)\Big|_0^{\pi} = 0$$

$$n\geq 2:\quad a_n = \frac{2}{\pi}\int_0^{\pi}\sin x\cdot\cos(nx)\,dx \underset{\uparrow}{=} \frac{\cancel{2}}{\pi}\cdot\frac{1}{\cancel{2}}\int_0^{\pi}\sin((n+1)x)+\sin((1-n)x)\,dx =$$

defaktorizacija: $\sin\alpha\cdot\cos\beta = \frac{1}{2}\big(\sin(\alpha+\beta)+\sin(\alpha-\beta)\big)$

$$= \frac{1}{\pi}\int_0^{\pi}\sin((n+1)x)\,dx - \frac{1}{\pi}\int_0^{\pi}\sin((n-1)x)\,dx = -\frac{1}{\pi}\cdot\frac{1}{n+1}\cdot\cos((n+1)x)\Big|_0^{\pi} + \frac{1}{\pi}\cdot\frac{1}{n-1}\cdot\cos((n-1)x)\Big|_0^{\pi} =$$

$$= -\frac{1}{\pi}\cdot\frac{1}{n+1}\cdot\cos((n+1)\pi) + \frac{1}{\pi}\cdot\frac{1}{n+1} + \frac{1}{\pi}\cdot\frac{1}{n-1}\cdot\cos((n-1)\pi) - \frac{1}{\pi}\cdot\frac{1}{n-1} =$$

$$= -\frac{1}{\pi}\cdot\frac{1}{n+1}\cdot(-1)^{n+1} + \frac{1}{\pi}\cdot\frac{1}{n-1}\cdot(-1)^{n-1} + \frac{1}{\pi}\cdot\left(\frac{1}{n+1}-\frac{1}{n-1}\right) =$$

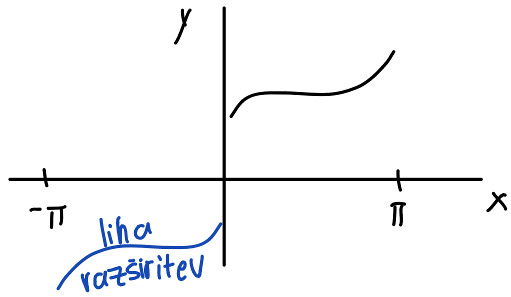
$$= \frac{1}{\pi}\cdot(-1)^n\cdot\left(\frac{1}{n+1}-\frac{1}{n-1}\right) + \frac{1}{\pi}\cdot\frac{\cancel{n}-1-\cancel{n}-1}{n^2-1} =$$

$$= \frac{(-1)^n}{\pi\cdot(n^2-1)}\cdot(\cancel{n}-1-\cancel{n}-1) + \frac{-2}{\pi(n^2-1)} = \frac{-2}{\pi(n^2-1)}\cdot\big((-1)^n+1\big)$$

$$\tilde{f}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) = \frac{2}{\pi} - \frac{2}{\pi}\cdot\sum_{n=2}^{\infty}\frac{(-1)^n+1}{n^2-1}\cos(nx)$$

SINUSNA FOURIERJEVA VRSTA

Naj bo f integrabilna na $(0,\pi)$.



$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot\sin(nx)\,dx$$

$$\tilde{f}(x) = \sum_{n=1}^{\infty} b_n\cdot\sin(nx)$$

Lastnosti:

- $\tilde{f}$ je liha
- $\tilde{f}$ je periodična s periodo 2π
- v točkah iz $(0,\pi)$, kjer je f zvezna: $\tilde{f}=f$
  kjer f ni zvezna: $\tilde{f}(x) = \frac{1}{2}\left(\lim\limits_{y\nearrow x} f(y) + \lim\limits_{y\searrow x} f(x)\right)$
- $\tilde{f}(0) = \tilde{f}(\pi) = 0$
- PARSEVALOVA ENAČBA: $\int_0^{\pi} (f(x))^2\,dx = \frac{\pi}{2}\sum_{n=1}^{\infty} b_n^2$

N: Funkcijo $f(x) = 1-x$ razvij na $[0,\pi]$ v Fourierjevo vrsto po samih sinusih.

$$b_n = \frac{2}{\pi}\int_0^{\pi}(1-x)\cdot\sin(nx)\,dx \overset{P.P.}{=} \frac{2}{\pi}\left((1-x)\cdot\left(-\cos(nx)\cdot\frac{1}{n}\right)\Big|_0^{\pi} - \int_0^{\pi}\frac{1}{n}\cos(nx)\,dx\right) =$$

$u = 1-x$ $\quad dv = \sin(nx)\,dx$

$du = -dx$ $\quad v = -\cos(nx)\cdot\frac{1}{n}$

$$= \frac{2}{\pi}\left((1-\pi)\cdot\frac{1}{n}\cdot(-\cos(n\pi)) + (1-0)\cdot\frac{1}{n}\cdot\cos 0\right) + \frac{1}{n^2}\sin(nx)\Big|_0^{\pi}\Big) =$$

$$= \frac{2}{\pi}\left((1-\pi)\cdot\frac{1}{n}\cdot(-1)^{n+1} + \frac{1}{n}\right) =$$

$$= \frac{2}{\pi n}\left((\pi-1)\cdot(-1)^n + 1\right)$$

$$\tilde{f}(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left((\pi-1)\cdot(-1)^n + 1\right)\cdot\sin(n\pi)$$

(N:) $f:[0,\pi]\to\mathbb{R}$, $f(x)=x\cdot(\pi-x)$ razvij v sinusovo Fourierjevo vrsto

Izračunaj $1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\ldots=\sum_{n=1}^{\infty}\frac{(-1)^n}{(2k-1)^3}$

Izračunaj $1+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\ldots=\sum_{n=1}^{\infty}\frac{1}{(2k-1)^6}$

$$b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cdot\sin(nx)\,dx=\frac{2}{\cancel{\pi}}\int_0^{\pi}\cancel{\pi}x\sin(nx)\,dx-\frac{2}{\pi}\int_0^{\pi}x^2\sin(nx)\,dx\overset{P.P.}{=}$$

$$=2\left(-\frac{x}{n}\cdot\cos(nx)\Big|_0^{\pi}+\frac{1}{n}\int_0^{\pi}\cos(nx)\,dx\right)-\frac{2}{\pi}\left(-\frac{x^2}{n}\cdot\cos(nx)\Big|_0^{\pi}+\frac{2}{n}\int_0^{\pi}x\cdot\cos(nx)\,dx\right)\overset{P.P.}{=}$$

$u=x$ $\quad dv=\sin(nx)dx$ $\qquad u=x^2$ $\quad dv=\sin(nx)dx$

$du=dx$ $\quad v=-\cos(nx)\cdot\frac{1}{n}$ $\qquad du=2xdx$ $\quad v=-\cos(nx)\cdot\frac{1}{n}$

$$=2\cdot\left(-\frac{\pi}{n}\cdot\cos(n\pi)+\frac{1}{n^2}\sin(nx)\Big|_0^{\pi}\right)-\frac{2}{\pi}\left(-\frac{\pi^2}{n}\cos(n\pi)+\frac{2}{n}\cdot\left(\frac{x}{n}\sin(nx)\Big|_0^{\pi}-\frac{1}{n}\cdot\int_0^{\pi}\sin(nx)\,dx\right)\right)=$$

$$=\cancel{-\frac{2\pi}{n}\cdot(-1)^n}+\cancel{\frac{2\pi}{n}\cdot(-1)^n}-\frac{4}{\pi n}\cdot\left(0+\frac{1}{n}\cdot\frac{1}{n}\cdot\cos(nx)\Big|_0^{\pi}\right)=$$

$u=x$ $\quad dv=\cos(nx)dx$

$du=dx$ $\quad v=\sin(nx)\cdot\frac{1}{n}$

$$=-\frac{4}{\pi n^3}\cdot\left(\cos(n\pi)-\cos 0\right)=-\frac{4}{\pi n^3}\cdot\left((-1)^n-1\right)=$$

$$=\frac{4}{\pi n^3}\cdot\left(1-(-1)^n\right)=\begin{cases}0; & n\text{ sod}\\ \frac{8}{\pi n^3}; & n\text{ lih}\end{cases}$$

$$\tilde{f}(x)=\sum_{n=1}^{\infty}\frac{4}{\pi n^3}\left(1-(-1)^n\right)\cdot\sin(nx)=\sum_{k=1}^{\infty}\frac{8}{\pi(2k-1)^3}\cdot\sin((2k-1)x)$$

$$x=\frac{\pi}{2}:\ \sin\left((2k-1)\cdot\frac{\pi}{2}\right)=\sin\left(\pi k-\frac{\pi}{2}\right)=-\cos(\pi k)=-(-1)^k=(-1)^{k+1}$$

$$\Rightarrow\ \frac{\pi}{2}\left(\pi-\frac{\pi}{2}\right)=f\left(\frac{\pi}{2}\right)=\tilde{f}\left(\frac{\pi}{2}\right)=\sum_{k=1}^{\infty}\frac{8}{\pi(2k-1)^3}\cdot(-1)^{k+1}$$

$$\frac{\pi^2}{4}=-\frac{8}{\pi}\cdot\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^3}\quad\Rightarrow\quad\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^3}=-\frac{\pi^3}{32}$$

$$\int_0^{\pi}|f(x)|^2dx=\frac{\pi}{2}\sum_{n=1}^{\infty}b_n^2\quad\Rightarrow\quad\int_0^{\pi}x^2(\pi-x)^2dx=\frac{\cancel{\pi}}{2}\cdot\sum_{n=1}^{\infty}\frac{64}{\pi^{\cancel{2}}(2n-1)^6}$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^6}=\frac{\pi}{32}\cdot\int_0^{\pi}\pi^2x^2-2\pi x^3+x^4\,dx=\frac{\pi}{32}\cdot\left(\pi^2\cdot\frac{x^3}{3}-2\pi\frac{x^4}{4}+\frac{x^5}{5}\right)\Big|_0^{\pi}=\frac{\pi^6}{32}\cdot\left(\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\right)=\frac{\pi^6}{32\cdot 30}$$