INTEGRACIJA PO DELIH (PER PARTES)

PRAVILO ZA ODVOD PRODUKTA:

$$(u\cdot v)' = u'\cdot v + u v'$$

ČE TO ZVEZO INTEGRIRAMO, DOBIMO ZVEZO:

$$u\cdot v = \int v u' dx + \int u v' dx .$$

ČE BI VPELJALI $u = u(x)$ IN $v = v(x)$, BI VELJALO:
$du = u' dx$ IN $dv = v' dx$. TOREJ JE:

$$u\cdot v = \int v\, du + \int u\, dv$$

$$\int u\, dv = u\cdot v - \int v\, du$$

INTEGRACIJA PO DELIH

PRIMER UPORABE:

$$\int x \cos x\, dx = \int u\, dv$$

IZBEREMO, KAJ OD TEGA JE u IN KAJ dv

NPR. $u = x$ $\quad dv = \cos x\, dx$

$du = u'\cdot dx = dx$ $\quad v = \int \cos x\, dx = \sin x + C$

BOŠ IZBEREMO EN KONKRETEN $C \in \mathbb{R}$ NPR. $C=0$.

PO FORMULI:

$$\int x\cos x\, dx = u\cdot v - \int v\, du =$$

$$= x \sin x - \int \sin x\, dx = x \sin x + \cos x + D$$

PREIZKUS: $(x\sin x + \cos x + D)' = \sin x + x\cos x - \sin x = x\cos x$ ✓

KLJUČNO VPRAŠANJE: KAKO IZBRATI u IN dv?

PAR OPAZK:

(i) dv MORA BITI TAK, DA GA ZNAMO INTEGRIRATI.

(ii) OBSTAJAJO FUNKCIJE, KI JIH ODVOD 'POENOSTAVI'.
NPR. $\ln x$, $\operatorname{arctg} x$, $\arcsin x$ ...
TAKE JE SMISELNO IZBRATI ZA u.

(iii) ČE JE $u = p(x)$, Z ODVAJANJEM ZNIŽAMO RED.
POLINOM

NALOGA: IZRAČUNAJ INTEGRALE:

(a) $\int \sin x \ln(\operatorname{tg} x)\,dx$ (b) $\int \frac{\arcsin x}{\sqrt{1+x}}\,dx$ (c) $\int \operatorname{arctg}\sqrt{x}\,dx$

REŠITEV:

(a) ZA u IZBEREMO LOGARITEM, KER RAČUNAMO, DA SE BO S TEM INTEGRAND POENOSTAVIL.

$u = \ln(\operatorname{tg} x)$ $\qquad$ $dv = \sin x\,dx$

$du = \frac{1}{\operatorname{tg} x} \cdot \frac{1}{\cos^2 x}\,dx$ $\qquad$ $v = -\cos x$

$du = \frac{\cancel{\cos x}}{\sin x} \cdot \frac{1}{\cos^{\cancel{2}} x}\,dx$

$$\int \sin x \ln(\operatorname{tg} x)\,dx = \overbrace{-\cos x \ln(\operatorname{tg} x)}^{u \cdot v} + \overbrace{\int \frac{dx}{\sin x}}^{\int v\,du} \cdot \frac{\sin x}{\sin x}$$

OD ZADNJIČ

$\boxed{t = \cos x \rightarrow dt = -\sin x\,dx}$

$$\int \sin x \ln(\operatorname{tg} x\,dx) = -\cos x \ln(\operatorname{tg} x) + \int \frac{dt}{t^2 - 1}$$

$$= -\cos x \cdot \ln(\operatorname{tg} x) + \frac{1}{2} \ln\left|\frac{\cos x - 1}{\cos x + 1}\right| + C$$

(b) $\displaystyle\int \frac{\arcsin x}{\sqrt{1+x}}\,dx$

IZBEREMO $u = \arcsin x$, KER TO POENOSTAVI INTEGRAND.

$u = \arcsin x$ $\qquad$ $dv = \dfrac{dx}{\sqrt{1+x}}$

$du = \dfrac{1}{\sqrt{1-x^2}}\,dx$ $\qquad$ $v = 2\sqrt{1+x}$

$$\int \frac{\arcsin x}{\sqrt{1+x}}\,dx = \underbrace{2\sqrt{1+x}\,\arcsin x}_{u\cdot v} - \underbrace{2\int \frac{dx}{\sqrt{1-x}}}_{\int v\,du}$$

$$= 2\sqrt{1+x}\,\arcsin x + 4\sqrt{1-x} + C$$

(c) $\displaystyle\int \operatorname{arctg}\sqrt{x}\,dx$

IZBEREMO $u = \operatorname{arctg}\sqrt{x}$, KER TO POENOSTAVI INTEGRAND.

$u = \operatorname{arctg}\sqrt{x}$ $\qquad$ $dv = dx$

$du = \dfrac{1}{1+x}\cdot\dfrac{1}{2\sqrt{x}}$ $\qquad$ $v = x$

$$\int \operatorname{arctg}\sqrt{x}\,dx = x\operatorname{arctg}\sqrt{x} - \frac{1}{2}\int \frac{\sqrt{x}}{1+x}\,dx$$

$$t = \sqrt{x} \Rightarrow dt = \frac{1}{2\sqrt{x}}\,dx = \frac{1}{2t}\,dx$$

$$= x\operatorname{arctg}\sqrt{x} - \frac{1}{2}\int \frac{t\cdot 2t\,dt}{1+t^2} =$$

$$= x\operatorname{arctg}\sqrt{x} - \int \frac{t^2+1-1}{1+t^2}\,dt =$$

$$= x\operatorname{arctg}\sqrt{x} - t + \operatorname{arctg} t + C =$$

$$= x\operatorname{arctg}\sqrt{x} - \sqrt{x} + \operatorname{arctg}\sqrt{x} + C$$

BOLJ OPTIMALNA POT:

$$u = \operatorname{arctg}\sqrt{x} \qquad dv = dx$$

$$du = \frac{1}{1+x}\cdot\frac{1}{2\sqrt{x}} \qquad \boxed{v = x+1}$$

VZELI SMO $C=1$

$$\int \operatorname{arctg}\sqrt{x}\,dx = (x+1)\operatorname{arctg}\sqrt{x} - \frac{1}{2}\int\frac{dx}{\sqrt{x}} =$$

$$= (x+1)\operatorname{arctg}\sqrt{x} - \sqrt{x} + C$$

DOBIMO ISTI REZULTAT.

NALOGA: IZRAČUNAJ INTEGRALA

$$\int P(x)\ln x\,dx \quad \text{IN} \quad \int P(x)\,e^{x}\,dx,$$

KJER JE $P(x)$ POLJUBEN POLINOM.

REŠITEV:

PIŠIMO $P(x) = a_n x^n + \ldots + a_1 x + a_0 = \sum_{k=0}^{n} a_k x^k.$

ZARADI LINEARNOSTI INTEGRALA JE DOVOLJ IZRAČUNATI INTEGRALA LE ZA x^k, $k \in \mathbb{N}_0$.

$$\int x^k \ln x\,dx = \frac{1}{k+1}x^{k+1}\cdot\ln x - \frac{1}{k+1}\int x^k\,dx =$$

$$u = \ln x \qquad dv = x^k\,dx$$

$$du = \frac{dx}{x} \qquad v = \frac{x^{k+1}}{k+1}$$

$$= \frac{1}{k+1}x^{k+1}\left(\ln x - \frac{1}{k+1}\right) + C$$

$$\int x^k e^x \, dx = x^k e^x - k\int x^{k-1} e^x \, dx$$

$u = x^k \qquad dv = e^x dx$

$du = k x^{k-1} dx \qquad v = e^x$

TAKTIKA JE ZNIŽATI RED POLINOMA

SEDAJ NADALJUJEMO 'PO INDUKCIJI':

$$\int x^k e^x \, dx = x^k e^x - k\left(x^{k-1} e^x - (k-1)\int x^{k-2} e^x dx\right) =$$

$$= x^k e^x - k x^{k-1} e^x + k(k-1) x^{k-2} e^x - \ldots \pm k! e^x + C$$

SKLEP:

$$\int P(x) \ln x \, dx = x\left(Q(x) + R(x)\ln x\right) + C,$$

$$\int P(x) e^x dx = S(x) e^x + C,$$

KJER SO Q, R IN S USTREZNI POLINOMI ISTE STOPNJE KOT POLINOM P.

NALOGA: DVAKRAT INTEGRIRAJ PO DELIH IN IZRAČUNAJ

$$I = \int e^{ax} \cos(bx) \, dx$$

REŠITEV: $u = e^{ax} \qquad dv = \cos(bx) \, dx$

$du = a e^{ax} dx \qquad v = \frac{\sin(bx)}{b}$

$$I = \frac{1}{b} e^{ax} \sin(bx) - \frac{a}{b}\int e^{ax} \sin(bx) dx$$

$u = e^{ax} \qquad dv = \sin(bx) dx$

$du = a e^{ax} dx \qquad v = -\frac{\cos(bx)}{b}$

$$I = \frac{1}{b} e^{ax} \sin(bx) - \frac{a}{b}\left(-\frac{1}{b} e^{ax}\cos(bx) + \frac{a}{b}\underbrace{\int e^{ax}\cos(bx)dx}\right)$$

TO JE NAŠ INTEGRAL

DOBIMO ENAČBO:

$$I = \frac{1}{b} e^{ax} \sin(bx) + \frac{a}{b^2} e^{ax}\cos(bx) - \frac{a^2}{b^2} I$$

$$I\left(1 + \frac{a^2}{b^2}\right) = \frac{1}{b} e^{ax}\left(\sin(bx) + \frac{a}{b}\cos(bx)\right)$$

$$I = \frac{b^2}{a^2+b^2} \cdot \frac{1}{b} e^{ax}\left(\sin(bx) + \frac{a}{b}\cos(bx)\right)$$

$$\boxed{I = \frac{e^{ax}}{a^2+b^2}\left(b\sin(bx) + a\cos(bx)\right) + C}$$

PODOBNO VELJA ZA $\int e^{ax}\sin(bx)dx$.