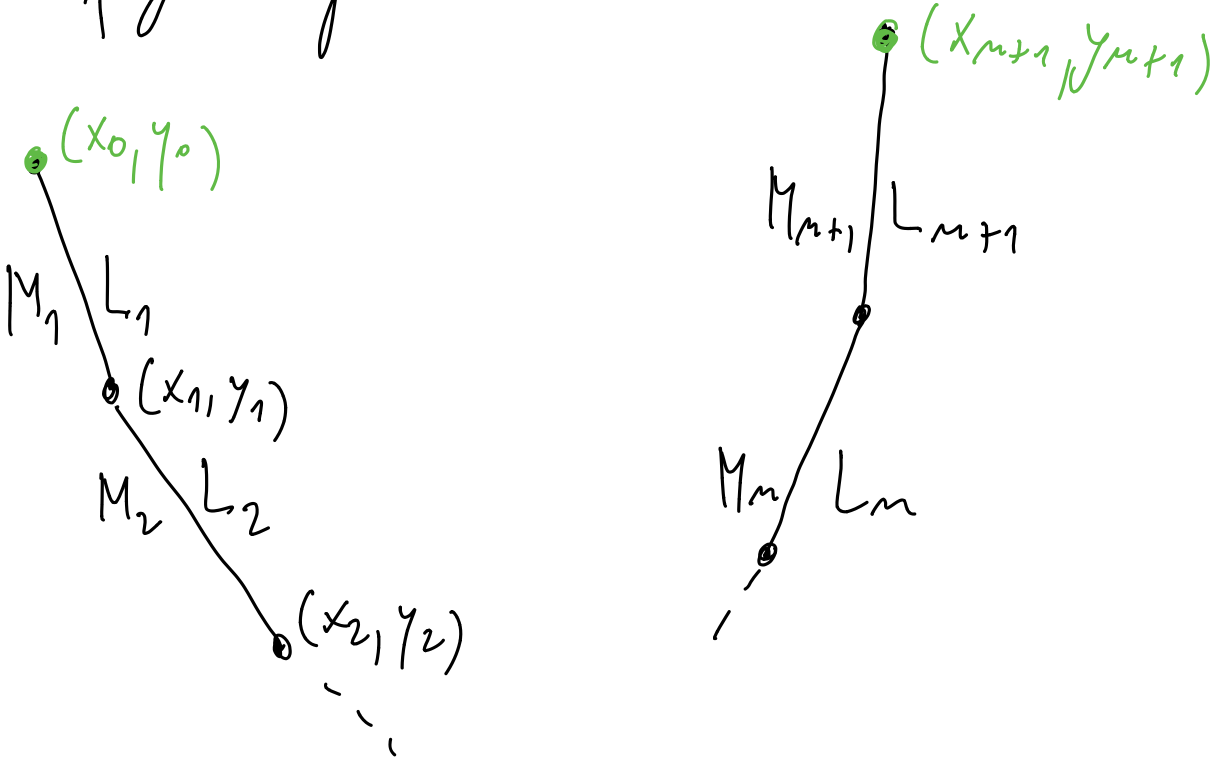
Diskretna verižnica

Imamo $n+1$ členkov (togih palic). Povežemo jih v celoto, pripnemo na dve obesišči in opazujemo, kakšna konfiguracija nastane.



L_i, $i = 1, 2, \dots, n+1$, so dolžine palic

M_i, $i = 1, 2, \dots, n+1$, so mase palic

(x_0, y_0) in (x_{n+1}, y_{n+1}) sta obesišči

Iščemo koordinate spojev palic

(x_i, y_i), $i = 1, 2, \dots, n$.

<u>Model</u>: Množica povezanih palic teži k minimizaciji potencialne energije (znižati želi težišče):

① Kakšna je potencialna energija i-te palice?

$$W_i = M_i g \frac{y_i + y_{i-1}}{2}, \quad i = 1, 2, \dots, n+1$$

② Skupna potencialna energija?

$$W = \sum_{i=1}^{n+1} M_i g \frac{y_i + y_{i-1}}{2}$$

Ker je g konstanta, lahko funkcijo poenostavimo

$$F(\underbrace{x}_{(x_0, \dots, x_{n+1})}, \underbrace{y}_{(y_0, \dots, y_{n+1})}) = \sum_{i=1}^{n+1} M_i \frac{y_{i-1} + y_i}{2}$$

Išćemo $\min\limits_{(x,y)} F(x,y)$, pri pogoju,

da je $(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2 = L_i^2$, $i = 1,2,\dots,n+1$

Rešujemo vezan ekstrem:

$$G(x,y,\lambda) = \sum_{i=1}^{n+1} M_i \frac{y_{i-1} + y_i}{2}$$

$$+ \lambda_i \left((x_i - x_{i-1})^2 + (y_i - y_{i-1})^2 - L_i^2 \right)$$

Poiskati moramo ekstrem funkcije G:

$$\frac{1}{2} \frac{\partial G}{\partial x_i} = 0, \quad i = 1,2,\dots,n,$$

$$\frac{1}{2} \frac{\partial G}{\partial y_i} = 0, \quad i = 1,2,\dots,n,$$

$$\frac{\partial G}{\partial \lambda_i} = 0, \quad i = 1,2,\dots,n+1.$$

$$\frac{1}{2} \frac{\partial G}{\partial x_i} \Rightarrow \lambda_i (x_i - x_{i-1}) - \lambda_{i+1} (x_{i+1} - x_i) = 0, \quad i = 1,\dots,n,$$

$$\frac{1}{2}\frac{\partial G}{\partial y_i} \Rightarrow \lambda_i(y_i - y_{i-1}) - \lambda_{i+1}(y_{i+1} - y_i) = -\frac{M_i + M_{i+1}}{4}$$

$$\frac{\partial G}{\partial \lambda_i} \Rightarrow (x - x_{i-1})^2 + (y_i - y_{i-1})^2 = L_i^2, \quad i = 1, 2, \dots, m+1.$$

Dobili smo sistem $3m+1$ nelinearnih enačb za $3m+1$ neznank.

Uvedemo relativne koordinate:

$$\xi_i = x_i - x_{i-1}, \quad i = 1, 2, \dots, m+1,$$

$$\eta_i = y_i - y_{i-1}, \quad i = 1, 2, \dots, m+1.$$

Torej je

$$x_i = x_0 + \sum_{j=1}^{i} \xi_j, \quad i = 1, 2, \dots, m+1,$$

$$y_i = y_0 + \sum_{j=1}^{i} \eta_j, \quad i = 1, 2, \dots, m+1.$$

Prejšnji sistem se prepiše v obliko

$$\lambda_i \xi_i - \lambda_{i+1}\xi_{i+1} = 0, \quad i=1,2,\dots,n$$

$$\lambda_i \eta_i - \lambda_{i+1}\eta_{i+1} = -\frac{1}{2}\mu_i, \quad i=1,2,\dots,n,$$

$$\xi_i^2 + \eta_i^2 = L_i^2, \quad i=1,2,\dots,n+1,$$

kjer je $\mu_i = \frac{1}{2}(m_i + m_{i+1})$.

$$\lambda_1\xi_1 = \lambda_2\xi_2 = \lambda_3\xi_3 = \dots = \lambda_{n+1}\xi_{n+1}$$

$$\Rightarrow \quad \lambda_i\xi_i = -\frac{1}{2u}, \quad i=1,2,\dots,n,$$ kjer

je u neznanka.

Iz zgornjih enačb sledi

$$\frac{1}{2u}\frac{\eta_i}{\xi_i} - \frac{1}{2u}\frac{\eta_{i+1}}{\xi_{i+1}} = \frac{1}{2}\mu_i, \quad i=1,2,\dots,n.$$

oz.

$$\frac{\eta_{i+1}}{\xi_{i+1}} = \frac{\eta_i}{\xi_i} - u\mu_i.$$

Iz zgornjih enačb izpeljemo:

$$\frac{\eta_i}{\xi_i} = \frac{\eta_{i-1}}{\xi_{i-1}} - \mu\mu_{i-1}$$

$$= \frac{\eta_{i-2}}{\xi_{i-2}} - \mu\mu_{i-2} - \mu\mu_{i-1}$$

$$\vdots$$

$$= \underbrace{\frac{\eta_1}{\xi_1}}_{=:v} - \mu\sum_{j=1}^{i-1}\mu_j$$

Od tod lahko izpeljemo:

$$\xi_i^2 + \eta_i^2 = L_i^2 \quad / : \xi_i^2$$

$$1 + \left(\frac{\eta_i}{\xi_i}\right)^2 = \frac{L_i^2}{\xi_i^2}$$

$$\xi_i = \frac{L_i}{\sqrt{1+\left(v - \mu\sum_{j=1}^{i-1}\mu_j\right)^2}}, \quad i = 1, 2, \ldots, n+1$$

Relativne koordinate η_i dobimo iz

zveze $\dfrac{\eta_i}{\xi_i} = v - u \sum_{j=1}^{i-1} \mu_j$, torej

$$\eta_i = \xi_i \left(v - u \sum_{j=1}^{i-1} \mu_j \right).$$

Enačbi za u in v dobimo

iz:
$$x_{n+1} = x_0 + \sum_{j=1}^{n+1} \xi_j(u,v) \quad \Rightarrow$$

$$y_{n+1} = y_0 + \sum_{j=1}^{n+1} \eta_j(u,v)$$

$$\boxed{\begin{aligned} U(u,v) &= \sum_{j=1}^{n+1} \xi_j(u,v) - (x_{n+1} - x_0) = 0 \\ V(u,v) &= \sum_{j=1}^{n+1} \eta_j(u,v) - (y_{n+1} - y_0) = 0 \end{aligned}}$$

Sistem dveh (nelinearnih) enačb za u in v!!!

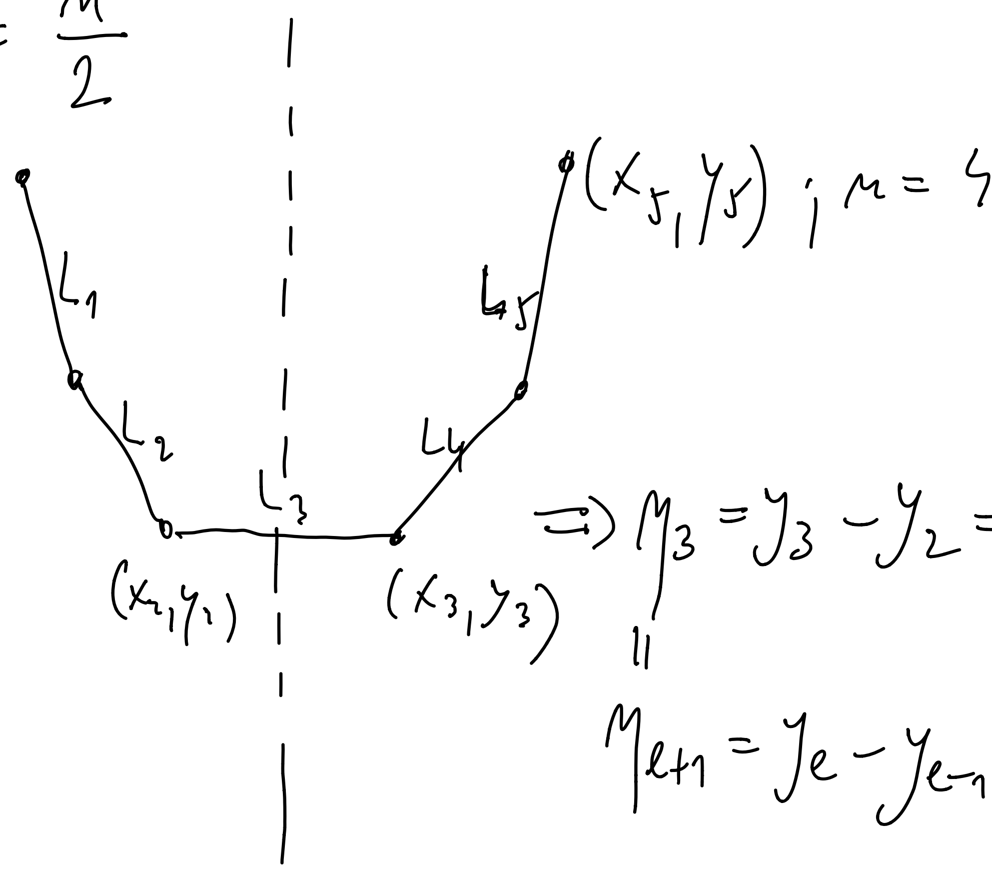
Simetrična verižnica z liho mnogo členki

① $y_0 = y_{n+1}$

② $\underbrace{L_1 = L_{n+1},\ L_2 = L_n, \dots;\ L_i = L_{n+2-i}}_{\text{pogoj za simetričnost}}$

$n+1 = 2l+1 \dots$ liho mnogo členkov

$l = \frac{n}{2}$



$\Rightarrow \eta_3 = y_3 - y_2 = 0$

$\|$

$\eta_{l+1} = y_l - y_{l-1} = 0$

$$\eta_{l+1} = \xi_{l+1}\left(\nu - \mu \sum_{j=1}^{l} \rho_j\right) = 0 \;/ : \xi_{l+1} \neq 0$$

$$\boxed{\nu = \mu \sum_{j=1}^{l} \rho_j}$$

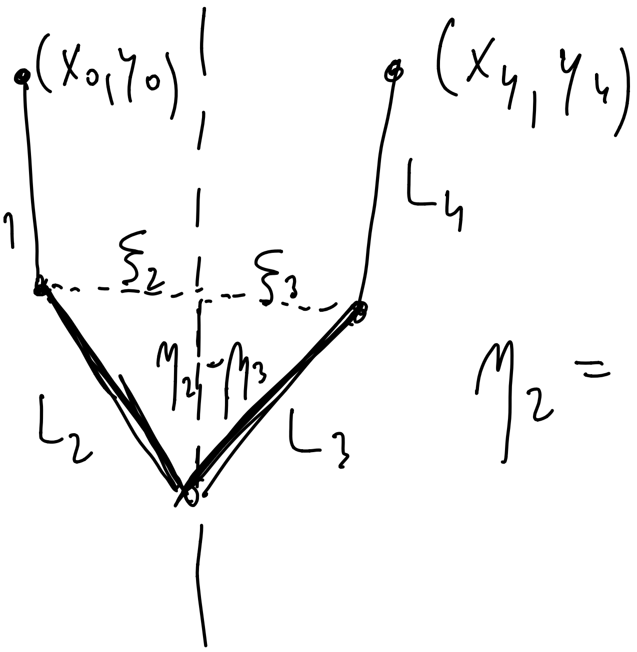
$$\Rightarrow \; U(\mu, \nu) = f(\mu) = 0 \; \Rightarrow$$

ena enačba z eno neznanko.

Simetrična verižnica s sodo mnogo členki

① $y_0 = y_{n+1}$

② $L_i = L_{n+2-i}, \; i = 1, 2, \dots \frac{n+1}{2}$



$n + 1 = 2l$

$l = \frac{n+1}{2}$

$\eta_2 = -\eta_3$

Splošno: $\eta_\ell = -\eta_{\ell+1}$

$$\xi_\ell\left(v-u\sum_{j=1}^{\ell-1}\mu_j\right)_4 = -\xi_{\ell+1}\left(v-u\sum_{j=1}^{\ell}\mu_j\right)$$

$\xi_2 = \xi_3 \neq 0$

$$v-u\sum_{j=1}^{\ell-1}\mu_j = -v+u\sum_{j=1}^{\ell}\mu_j$$

$$v = \frac{u}{2}\left(\sum_{j=1}^{\ell-1}\mu_j + \sum_{j=1}^{\ell}\mu_j\right)$$

$$= u\left(\sum_{j=1}^{\ell-1}\mu_j + \frac{\mu_\ell}{2}\right)$$

$\Rightarrow$ ena enačba z eno neznanko.