- Naj bo A kompleksna $n \times n$ matrika.

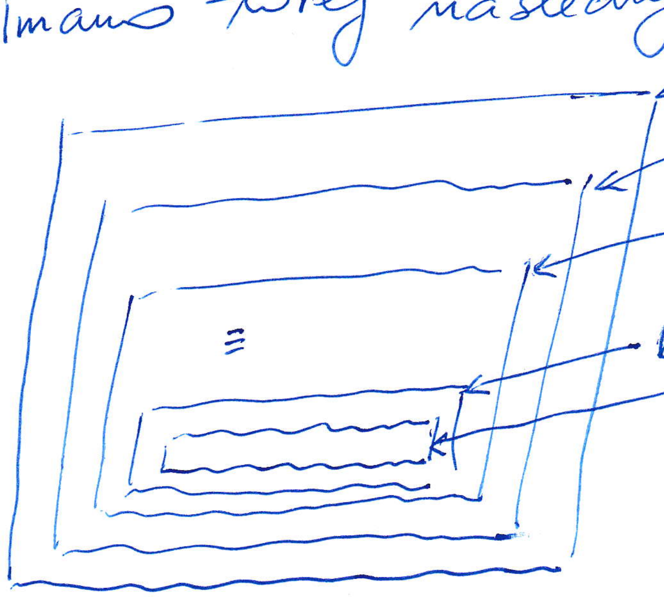
Iščemo tako obrnljivo matriko P in tako Jordansko matriko J, da je $A = PJP^{-1}$!

Naj bodo $\lambda_1, \dots, \lambda_k$ vse paroma različne lastne vrednosti matrike A. Poiskali bomo take matrike $P(\lambda_1), \dots, P(\lambda_k)$ in take matrike $J(\lambda_1), \dots, J(\lambda_k)$, da velja

$$P = [P(\lambda_1) \cdots P(\lambda_k)] \text{ in } J = \begin{bmatrix} J(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & J(\lambda_k) \end{bmatrix}$$

Naj bo λ lastna vrednost matrike A.
Označimo $N = A - \lambda I$.
Izračunamo potence $N, N^2, N^3, \dots$ in njihova jedra $\ker N \subseteq \ker N^2 \subseteq \ker N^3 \subseteq \dots$
Opazimo, da jedra nekaj časa naraščajo, nato pa se ustalijo. Poiščemo torej tak r, da velja $\ker N \subsetneq \ker N^2 \subsetneq \dots \subsetneq \ker N^r = \ker N^{r+1}$

Imamo torej naslednjo situacijo



$\ker N^r$ = korenski podprostor za lastno vrednost λ

$\ker N^{r-1}$

$\ker N^{r-2}$

$\vdots$

$\ker N^2$

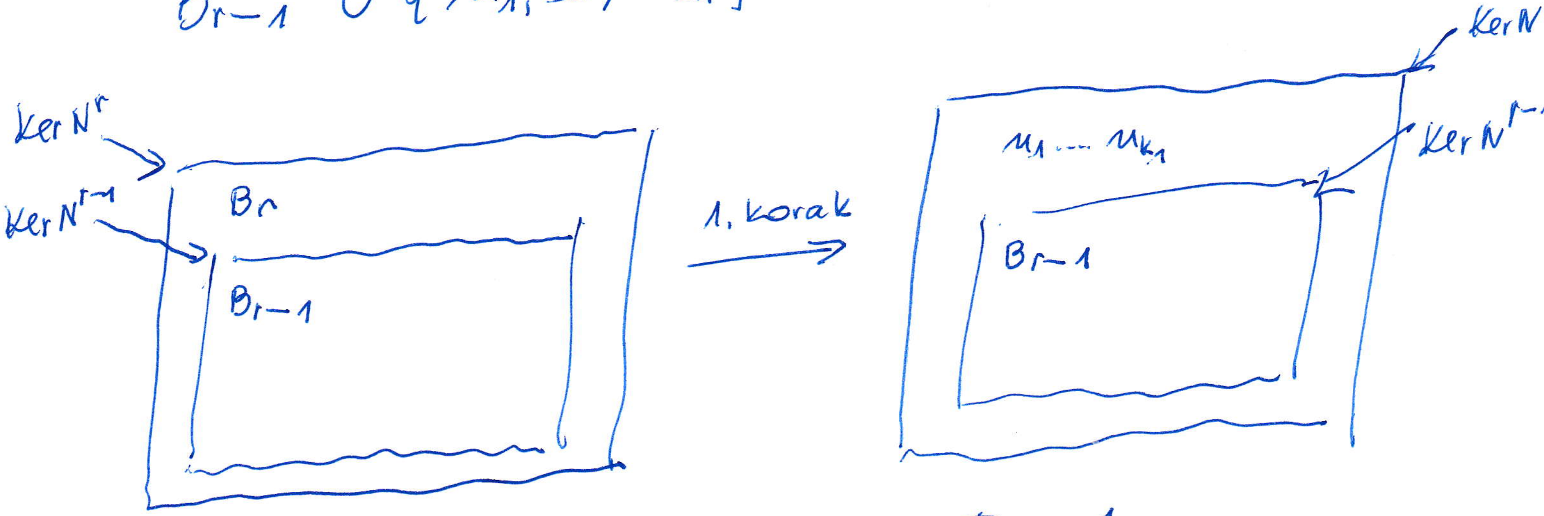
$\ker N$ = lastni podprostor za lastno vrednost λ

- Za vsak $i=1,\dots,r$ izberimo bazo B_i za podprostor $\ker N^i$. Tem bazam bomo rekli <u>povežne baze</u>.

Konstrukcija jordanske baze za $\ker N^r$

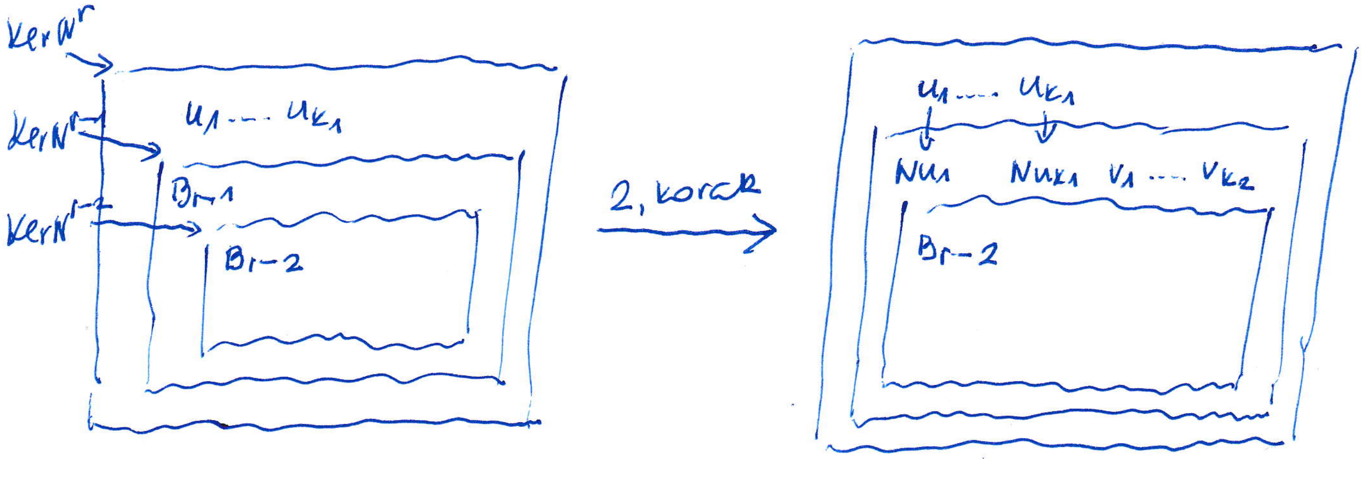
1. korak Popravljanje baze B_r

Poiščimo take $u_1,\dots,u_{k_1} \in B_r$, da je $B_{r-1} \cup \{u_1,\dots,u_{k_1}\}$ baza za $\ker N^r$



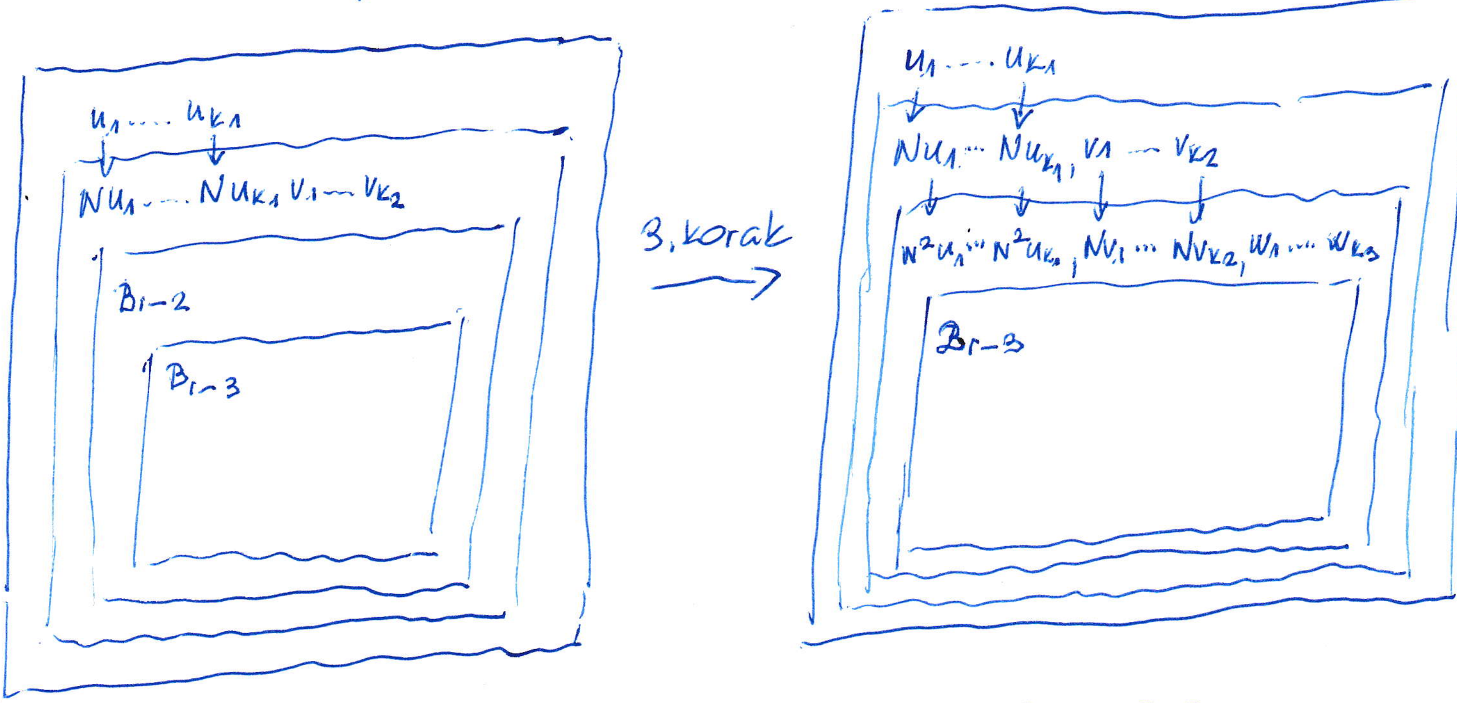
2. korak Popravljanje baze B_{r-1}

Poiščimo take $v_1,\dots,v_{k_2} \in B_{r-1}$, da je $B_{r-2} \cup \{Nu_1,\dots,Nu_{k_1}\} \cup \{v_1,\dots,v_{k_2}\}$ baza za $\ker N^{r-1}$



- 3. korak Popravljanje baze B_{r-2}

Poiščimo take $w_1, \dots, w_{k_3} \in B_{r-2}$, da je množica $B_{r-3} \cup \{N^2u_1, \dots, N^2u_{k_1}\} \cup \{Nv_1, \dots, Nv_{k_2}\} \cup \{w_1, \dots, w_{k_3}\}$ baza za podprostor $\ker N^{r-2}$



S tem postopkom nadaljujemo dokler gre.
Po r korakih dobimo naslednjo situacijo

$$
\begin{array}{llllll}
u_1, \dots, u_{k_1} & & & & & \\
\downarrow & & & & & \\
Nu_1, \dots, Nu_{k_1}, & v_1, \dots, v_{k_2} & & & & \\
\downarrow & \downarrow & & & & \\
N^2u_1, \dots, N^2u_{k_1}, & Nv_1, \dots, Nv_{k_2}, & w_1, \dots, w_{k_3} & & & \\
\downarrow & \downarrow & \downarrow & & & \\
\vdots & \vdots & \vdots & & & \\
\downarrow & \downarrow & \downarrow & \downarrow \dots \downarrow & & \\
N^{r-2}u_1, \dots, N^{r-2}u_{k_1}, & N^{r-3}v_1, \dots, N^{r-3}v_{k_2}, & N^{r-4}w_1, \dots, N^{r-4}w_{k_3}, & \dots, & t_1, \dots, t_{k_{r-1}} & \\
\downarrow & \downarrow & \downarrow & & \downarrow & \\
N^{r-1}u_1, \dots, N^{r-1}u_{k_1}, & N^{r-2}v_1, \dots, N^{r-2}v_{k_2}, & N^{r-3}w_1, \dots, N^{r-3}w_{k_3}, & \dots, & Nt_1, \dots, Nt_{k_{r-1}}, & z_1, \dots, z_{k_r}
\end{array}
$$

- Imamo torej

k_1 jordanskih verig dolžine r

k_2 -||- $r-1$

k_3 -||- $r-2$

$\vdots$

k_{r-1} -||- 2

k_r -||- 1

Število vseh verig se ujema z dimenzijo lastnega podprostora $k_1+\dots+k_r = \dim \ker N$

Del matrike P, ki pripada lastni vrednosti λ je torej

$$P(\lambda) = \Big[\underbrace{N^{r-1}u_1, N^{r-2}u_1, \dots, N^2u_1, Nu_1, u_1}_{\text{prva j. veriga dolžine } r}, \dots, \underbrace{N^{r-1}u_{k_1}, N^{r-2}u_{k_1}, \dots, N^2u_{k_1}, Nu_{k_1}, u_{k_1}}_{\text{zadnja j. veriga dolžine } r},$$

$$\underbrace{N^{r-2}v_1, \dots, v_1}_{\text{prva j. veriga dolžine } r-1}, \dots, \underbrace{N^{r-2}v_{k_2}, \dots, v_{k_2}}_{\text{zadnja j. veriga dolžine } r-1}, \underbrace{N^{r-3}w_1, \dots, w_1}_{\text{prva j. veriga dolžine } r-2}, \dots, \underbrace{N^{r-3}w_{k_3}, \dots, w_{k_3}}_{\text{zadnja j. veriga dolžine } r-2}$$

$$\dots, \underbrace{Nt_1, t_1}_{\text{prva j. veriga dolžine } 2}, \dots, \underbrace{Nt_{k_{r-1}}, t_{k_{r-1}}}_{\text{zadnja j. veriga dolžine } 2}, \underbrace{z_1}_{\text{prva j. veriga dolžine } 1}, \dots, \underbrace{z_{k_r}}_{\text{zadnja j. veriga dolžine } 1}\Big]$$

Celotna prehodna matrika je

$$P = [P(\lambda_1), \dots, P(\lambda_k)]$$

Mi smo izračunali enega od teh k-delov.
Za ostale dele je enak postopek.

- Označimo

$$J_s(\lambda) = \underbrace{\begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}}_{s \text{ stolpcev}} \Big\} \, s \text{ vrstic}$$

Del matrike J, ki pripada lastni vrednosti λ je torej

$$J(\lambda) = \begin{bmatrix} \underbrace{J_r(\lambda) \ddots J_r(\lambda)}_{k_1\text{-krat}} & & & & \\ & \underbrace{J_{r-1}(\lambda) \ddots J_{r-1}(\lambda)}_{k_2\text{-krat}} & & & \\ & & \underbrace{J_{r-2}(\lambda) \ddots J_{r-2}(\lambda)}_{k_3\text{-krat}} & & \\ & & & \ddots & \\ & & & & \underbrace{J_2(\lambda) \ddots J_2(\lambda)}_{k_{r-1}\text{ krat}} \\ & & & & & \underbrace{J_1(\lambda) \ddots J_1(\lambda)}_{k_r\text{ krat}} \end{bmatrix}$$

Celotna jordanska matrika je

$$J = \begin{bmatrix} J(\lambda_1) & & \\ & \ddots & \\ & & J(\lambda_k) \end{bmatrix}$$

Mi smo izračunali samo enega od teh k delov.
Za ostale dele je enak postopek

Primer: Naj bo $A = \begin{bmatrix} 0 & 1 & -1 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

$\det(A - xI) = x(x-2)^3$

lastni vrednosti sta 0 in 2

Ustrezna korenska podprostora sta dimenzij 1 in 3

Najprej izračunamo pomožne baze za

$\ker A$, $\ker(A-2I)$, $\ker(A-2I)^2$, $\ker(A-2I)^3$

$\ker A$: $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\ker(A-2I)^2 = \ker(A-2I)^3$: $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$

$\ker(A-2I)$: $\begin{bmatrix} 3 \\ 0 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$

To je že tisti del jordanske baze, ki pripada l.v. 0

popravimo B_2

$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 3 \\ 0 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$

popravimo B_1

$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}$

$\Big\downarrow A-2I$

$\begin{bmatrix} 2 \\ 4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \\ 2 \end{bmatrix}$

To je tisti del jordanske baze, ki pripada l.v. 2

· Velja torej

$$P = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

j. veriga za 0 j. veriga za 2 j. veriga za 2

· in

$$J = \begin{bmatrix} \boxed{0} & & & \\ & 2 & 1 & \\ & & 2 & \\ & & & \boxed{2} \end{bmatrix}$$

j. kletka za 0 j. kletke za 2

velikosti kletk se ujemajo z dolžinami jordanskih verig

Kako potenciramo jordansko kletko?

$$\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}^3 = ?$$

$$\begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda & & & \\ & \lambda & & \\ & & \lambda & \\ & & & \lambda \end{bmatrix}}_{\lambda I} + \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{N}$$

$$(\lambda I) \cdot N = N \cdot (\lambda I)$$

Lahko uporabimo formulo za potenco binoma

$$(a+b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3$$

$$(\lambda I + N)^3 = (\lambda I)^3 + 3(\lambda I)^2 N + 3(\lambda I) N^2 + N^3$$

$$= \lambda^3 I + 3\lambda^2 N + 3\lambda N^2 + N^3$$

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad N^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad N^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda^3 & 3\lambda^2 & 3\lambda & 1 \\ & \lambda^3 & 3\lambda^2 & 3\lambda \\ & & \lambda^3 & 3\lambda^2 \\ & & & \lambda^3 \end{bmatrix}$$

- Podobno

$$(\lambda I + N)^4 = \lambda^4 I + 4\lambda^3 \underset{\begin{bmatrix} 0&1&0&0\\ &&1&\\ &&&1\\ &&& \end{bmatrix}}{\underset{\shortparallel}{N}} + 6\lambda^2 \underset{\begin{bmatrix} 0&0&1&0\\ &&&1\\ &&&\\ &&& \end{bmatrix}}{\underset{\shortparallel}{N^2}} + 4\lambda \underset{\begin{bmatrix} 0&0&0&1\\ &&&\\ &&&\\ &&& \end{bmatrix}}{\underset{\shortparallel}{N^3}} + \underset{0}{\underset{\shortparallel}{N^4}}$$

$$= \begin{bmatrix} \lambda^4 & 4\lambda^3 & 6\lambda^2 & 4\lambda \\ & \lambda^4 & 4\lambda^3 & 6\lambda^2 \\ & & \lambda^4 & 4\lambda^3 \\ & & & \lambda^4 \end{bmatrix}$$