Integracija kotnih funkcij

Uros KUZMAN

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Trik (1)

Imejmo integral oblike $\int \sin^n x \cos^m x dx$, $n, m \in \mathbb{Z}$.

- *Če je* n *lih, vpeljemo* $t = \cos x$.
- *Če je* m *lih, vpeljemo* $t = \sin x$.
- *Če sta obe potenci sodi, uporabimo zvezi*

$$\cos^2 x = \frac{1 + \cos 2x}{2}, \quad \sin^2 x = \frac{1 - \cos 2x}{2}.$$

Primer:

$$I = \int \sin^{10} x \cos^3 x \; dx.$$

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$$dt = \cos x \; dx \Longrightarrow dx = \frac{dt}{\cos x}.$$

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Torej je

$$I = \int t^{10} \cos^2 x dt =$$

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Torej je

$$I = \int t^{10} \cos^2 x dt = \int t^{10} (1 - \sin^2 x) dt = \int t^{10} - t^{12} dt =$$

$$= \frac{1}{11} t^{11} - \frac{1}{13} t^{13} + C = \frac{1}{11} \sin^{11} x - \frac{1}{13} \sin^{13} x + C.$$

Primer:

$$I = \int \cos^4 x \sin^2 x dx.$$

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Ker sta obe potenci sodi, lahko uporabimo zvezi

$$\cos^4 x \sin^2 x = \left(\frac{1+\cos 2x}{2}\right)^2 \left(\frac{1-\cos 2x}{2}\right) =$$

Primer:

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$$\cos^4 x \sin^2 x = \left(\frac{1+\cos 2x}{2}\right)^2 \left(\frac{1-\cos 2x}{2}\right) = \frac{1}{8}(1-\cos^2 2x)(1+\cos 2x).$$

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Dobimo vsoto integralov

$$I = \frac{1}{8}\int dx + \frac{1}{8}\int \cos 2x dx - \frac{1}{8}\int \cos^2 2x dx - \frac{1}{8}\int \cos^3 2x dx.$$

Primer:

$$I = \int \cos^4 x \sin^2 x dx.$$

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Integracija prvih dveh sumandov je enostavna

$$I = \frac{1}{8}x + \frac{1}{16}\sin 2x - I_1 - I_2.$$

Zato se posvetimo preostalima dvema.

Pri prvem znova uporabimo izraz za kvadrat kosinusa

$$I_1 = \frac{1}{8} \int \cos^2 2x dx =$$

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Pri drugem uvedemo $t = \sin 2x$ oz. $dt = 2\cos 2x dx$

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$$I_2 = \frac{1}{8}\int \cos^3 2x dx = \frac{1}{16}\int (1-t^2)dt = \frac{1}{16}(\sin 2x - \frac{\sin^3 2x}{3}) + C.$$

Primer:

$$I = \int \sin 6x \cos 13x dx.$$

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Uporabimo zvezo

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Imamo torej

$$I = \frac{1}{2} \int \sin 19x - \sin 7x dx = -\frac{1}{38} \cos 19x + \frac{1}{14} \cos 7x + C.$$

Trik (2)

Naj bo R *racionalna funkcija. Integrale oblike* $\int R(\sin x, \cos x)dx$ *rešujemo s pomočjo unverzalne substitucije* $t = \tan\frac{x}{2}$*, za katero velja:*

$$dx = \frac{2dt}{1+t^2},\ \cos x = \frac{1-t^2}{1+t^2},\ \sin x = \frac{2t}{1+t^2}.$$

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$$t = \tan\frac{x}{2} \Longrightarrow \cos^2\frac{x}{2} = \frac{1}{1+\tan^2\frac{x}{2}} = \frac{1}{1+t^2}.$$

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Od tod sledi

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$$dt = \frac{dx}{2\cos^2\frac{x}{2}} \Longrightarrow dx = \frac{2dt}{1+t^2}.$$

$$\cos x = \cos 2\left(\frac{x}{2}\right) = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} =$$

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$$\sin x = \sqrt{1-\cos^2 x} =$$

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$$\sin x = \sqrt{1-\cos^2 x} = \sqrt{1 - \left(\frac{1-t^2}{1+t^2}\right)^2}$$

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$$\sin x = \sqrt{1-\cos^2 x} = \sqrt{1-\left(\frac{1-t^2}{1+t^2}\right)^2} = \sqrt{\frac{4t^2}{(1+t^2)^2}} = \frac{2t}{1+t^2}.$$

Primer:

$$I = \int \frac{1}{5 + 4\cos x} dx.$$

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Uporabimo substitucijo $t = \tan \frac{x}{2}$. Dobimo

$$I = \int \frac{1}{5 + 4 \cdot \frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2} =$$

Primer:

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Uporabimo substitucijo $t = \tan \frac{x}{2}$. Dobimo

$$I = \int \frac{1}{5 + 4 \cdot \frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2} = \int \frac{2dt}{5(1+t^2) + 4(1-t^2)} = \int \frac{2dt}{t^2+9}.$$

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Ta integral prevedemo na integral funkcije arctan z uvedbo $s = \frac{t}{3}$

$$I = \int \frac{2}{9} \frac{dt}{1 + \left(\frac{t}{3}\right)^2} =$$

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Uporabimo substitucijo $t = \tan \frac{x}{2}$. Dobimo

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Primer:

$$I = \int \frac{1}{5 + 4\cos x} dx.$$

Uporabimo substitucijo $t = \tan \frac{x}{2}$. Dobimo

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$$I = \int \frac{2}{9} \frac{dt}{1 + \left(\frac{t}{3}\right)^2} = \frac{2}{3} \int \frac{ds}{1+s^2} = \frac{2}{3} \arctan\left(\frac{1}{3} \tan \frac{x}{2}\right) + C.$$

Trik (3)

Naj bo P polinom. Integrale oblike $\int P(x)\sin x dx$ in $\int P(x)\cos x dx$ računamo z integracijo po delih. Natančneje, izberemo $u = P(x)$ in z odvajanjem nižamo stopnjo polioma.

Primer:

$$I = \int x^2 \sin x dx.$$

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Izberemo

$$u = x^2, \quad dv = \sin x dx,$$

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Izberemo

$$u = x^2, \quad dv = \sin x dx,$$

$$du = 2x dx, \quad v = -\cos x.$$

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$$I = \int x^2 \sin x dx.$$

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$$u = x^2, \quad dv = \sin x dx,$$

$$du = 2x dx, \quad v = -\cos x.$$

Dobimo

$$I = -x^2 \cos x + 2 \int x \cos x dx.$$

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$$u = x^2, \quad dv = \sin x dx,$$

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Dobimo

$$I = -x^2 \cos x + 2 \int x \cos x dx.$$

Postopek ponovimo z

$$u = x, \quad dv = \cos x dx,$$

Primer:

$$I = \int x^2 \sin x dx.$$

Izberemo

$$u = x^2, \quad dv = \sin x dx,$$

$$du = 2x dx, \quad v = -\cos x.$$

Dobimo

$$I = -x^2 \cos x + 2\int x \cos x dx.$$

Postopek ponovimo z

$$u = x, \quad dv = \cos x dx,$$

$$du = dx, \quad v = \sin x.$$

Primer:

$$I = \int x^2 \sin x dx.$$

Izberemo

$$u = x^2, \quad dv = \sin x dx,$$

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Dobimo

$$I = -x^2 \cos x + 2 \int x \cos x dx.$$

Postopek ponovimo z

$$u = x, \quad dv = \cos x dx,$$

$$du = dx, \quad v = \sin x.$$

Dobimo

$$I = -x^2 \cos x + 2x \sin x - 2 \int \sin x dx =$$

Primer:

$$I = \int x^2 \sin x dx.$$

Izberemo

$$u = x^2, \quad dv = \sin x dx,$$

$$du = 2x dx, \quad v = -\cos x.$$

Dobimo

$$I = -x^2 \cos x + 2 \int x \cos x dx.$$

Postopek ponovimo z

$$u = x, \quad dv = \cos x dx,$$

$$du = dx, \quad v = \sin x.$$

Dobimo

$$I = -x^2 \cos x + 2x \sin x - 2 \int \sin x dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C.$$