PRIMERI ZA VAJO

1. INTEGRIRAJ RACIONALNE FUNKCIJE:

(a) $\int \frac{21x^5}{x^6+9x^3+8}\,dx$ (b) $\int \frac{2x+1}{x^2-2x+2}\,dx$

(c) $\int \frac{2\,dx}{(x^2+1)^2}$

2. INTEGRIRAJ IRACIONALNE FUNKCIJE:

(a) $\int \frac{2+\sqrt{x+1}}{x+1-\sqrt{x+1}}\,dx$ (b) $\int \frac{dx}{\sqrt{-x^2+2x+3}}$

(c) $\int \frac{dx}{x^2\sqrt{1-x^2}}$

3. INTEGRIRAJ KOTNE OZ. KROŽNE FUNKCIJE:

(a) $\int \cos^5 x\,dx$ (b) $\int \frac{dx}{1+\sin^2 x}$

(c) $\int \arccos\sqrt{\frac{x}{x+1}}\,dx$ (d) $\int \arcsin^3 x\,dx$.

4. INTEGRIRAJ LOGARITEMSKE OZ. EKSPONENTNE FUNKCIJE:

(a) $\int x^2 e^{3x}\,dx$ (b) $\int \ln^2 x\,dx$

(c) $\int \frac{\ln(\ln x)}{x}\,dx$

REŠITVE:

1 (a)

$$I=\int \frac{21x^5}{x^6+9x^3+8}\,dx$$

DIREKTNA POT:

$$(x^6+9x^3+8)=(x^3+1)(x^3+8)=(x+1)(x^2-x+1)(x+2)(x^2-2x+4)$$

DOBIMO SISTEM S 6 KONSTANTAMI.

BOLJŠI PRISTOP:

$$t=x^3 \Rightarrow dt=3x^2dx \Rightarrow dx=\frac{dt}{3x^2}$$

$$I=\int \frac{7x^3}{t^2+9t+8}\,dt=\int\frac{7t}{(t+1)(t+8)}=$$

$$=A\cdot \ln|t+1|+B\ln|t+1|$$

PREVERI: $A=-1,\quad B=8$

$$I=-\ln|x^3+1|+8\ln|x^3+8|+C$$

(b) $$\int\frac{2x+1}{x^2-2x+2}\,dx=A\cdot\ln|x^2-2x+2|+B\operatorname{arctg}\frac{2x-2}{2}+C$$

$D=4-8=-4 \Rightarrow \sqrt{-D}=2$

PREVERI Z ODVAJANJEM: $A=1$ $\quad B=3$

(c) $$\int \frac{dx}{(x^2+1)^2} = A \ln(x^2+1) + B \operatorname{arctg} \frac{2x}{2} + \frac{Cx+\tilde{D}}{x^2+1} + E$$

$D = -4 \Rightarrow \sqrt{-D} = 2$

PREVERI Z ODVAJANJEM:

$A = 0, \quad B = 1, \quad C = 1, \quad \tilde{D} = 0$

(2) (a) $$\int \frac{2+\sqrt{x+1}}{x+1-\sqrt{x+1}}\, dx = \int \frac{(2+t)}{t^2-t} \cdot 2t\, dt =$$

$t = \sqrt{x+1} \Rightarrow dt = \frac{dx}{2\sqrt{x+1}} \Rightarrow dx = 2t\, dt$

$$= \int \frac{2t+4}{t-1}\, dt = \int \left(2 + \frac{6}{t-1}\right) dt = 2t + 6\ln|t-1| + C =$$

$$= 2\sqrt{x+1} + 6 \ln\left|\sqrt{x+1} - 1\right| + C$$

(b) $$I = \int \frac{dx}{\sqrt{-x^2+2x+3}}$$

$\leftarrow$ PREVEDENO NA arcsin, KER VODILNI ČLEN NEGATIVEN

$$-x^2+2x+3 = -(x^2-2x-3) = -((x-1)^2-4) = 4-(x-1)^2$$

$$I = \int \frac{dx}{\sqrt{4-(x-1)^2}} = \int \frac{dt}{\sqrt{4-t^2}} = \arcsin\left(\frac{t}{2}\right) + C$$

$t = x-1$
$dt = dx$

$$= \arcsin\left(\frac{x-1}{2}\right) + C$$

(c) $I = \int \frac{dx}{x^2\sqrt{1-x^2}}$ ← KER VODILNI ČLEN NEGATIVEN, UPORABIMO

$$\sqrt{-a}\,(x-x_1)\,u = \sqrt{ax^2+bx+c}$$

x_1 ← NIČLA

$x_1 = -1$ (LAHKO BI BILA TUDI $+1$)

$ax^2+bx+c = 1-x^2$

SUBSTITUCIJA:

$$(x+1)u = \sqrt{1-x^2}$$

$$u = \sqrt{\frac{1-x}{1+x}} \Rightarrow du = \frac{1}{2}\sqrt{\frac{1+x}{1-x}}\ \frac{-1-x-(1-x)}{(1+x)^2}dx$$

$$du = -\frac{1}{u}\ \frac{dx}{(1+x)^2}$$

$$I = \int \frac{-u(1+x)^2\,du}{x^2\underbrace{\sqrt{1-x^2}}_{=(x+1)u}} = -\int \frac{1+x}{x^2}\,du = \frac{\sqrt{1-x^2}}{u}$$

$$= -\int \frac{\sqrt{1-x^2}}{x^2\cdot u}\,du = -2\int \frac{\frac{u}{1+u^2}}{\frac{(1-u^2)^2}{(1+u^2)^2}\cdot u} = -2\int \frac{(1+u^2)}{(1-u^2)^2}\,du$$

$$u = \sqrt{\frac{1-x}{1+x}} \Rightarrow u^2\cdot(1+x) = 1-x$$

$$\Rightarrow x(u^2+1) = 1-u^2$$

$$x = \frac{1-u^2}{1+u^2}$$

$$x^2 = \frac{1-2u^2+u^4}{1+2u^2+u^4}$$

$$\sqrt{1-x^2} = \sqrt{\frac{4u^2}{1+2u^2+u^4}} = \frac{2u}{1+u^2}$$

INTEGRIRAJ DO KONCA SAM:

$$I = A\ln|u-1| + B\ln|u+1| + \frac{Cu+D}{(u-1)(u+1)} + E$$

ALTERNATIVNA POT: $t = \frac{1}{x} \Rightarrow I = -\int \frac{t\,dt}{\sqrt{t^2-1}}$

$s = t^2-1 \Rightarrow I = -\frac{1}{2}\int \frac{ds}{\sqrt{s}}$

3 (a) $\int \cos^3 x\,dx = \int \cos^4 x\,dt = \int (1-\sin^2 x)^2\,dt =$

$t = \sin x$

$dt = \cos x\,dx$

$$= \int (1-t^2)^2\,dt = t - \frac{2t^3}{3} + \frac{t^5}{5} + C$$

$$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$$

(b) $I = \int \frac{dx}{1+\sin^2 x} = \int \frac{dx}{1+\frac{1-\cos 2x}{2}} = \int \frac{2}{3-\cos 2x}\,dx$

$u = 2x \Rightarrow du = 2\,dx \Rightarrow I = \int \frac{du}{3-\cos u}$

$t = tg\,\frac{u}{2}$ $\qquad du = \frac{2dt}{1+t^2} \quad \cos u = \frac{1-t^2}{1+t^2} \quad \sin u = \frac{2t}{1+t^2}$

$$I = \int \frac{1}{3 - \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt = \int \frac{dt}{1+2t^2} =$$

$$= \int \frac{dt}{1+(\sqrt{2}t)^2} = \frac{1}{\sqrt{2}}\,arctg\,(\sqrt{2}t) + C =$$

$$= \frac{1}{\sqrt{2}}\,arctg\,(\sqrt{2}\,tg\,x) + C$$

(c) $$\int \arccos\sqrt{\frac{x}{x+1}}\,dx = x\arccos\sqrt{\frac{x}{x+1}} + \frac{1}{2}\int \frac{\sqrt{x}}{x+1}\,dx$$

PER PARTES

$u = \arccos\sqrt{\frac{x}{x+1}}$ $\qquad dv = dx$

$du = -\frac{dx}{2\sqrt{x}(x+1)}$ $\qquad v = x$

$t = \sqrt{x}$

$dt = \frac{dx}{2\sqrt{x}} = \frac{dx}{2t}$

$$= x\arccos\sqrt{\frac{x}{x+1}} + \int \frac{t^2}{t^2+1}\,dt$$

SMO ŽE IZRAČUNALI

$t - \operatorname{arctg} t + C$

(d) $$\int \arcsin^3 x\,dx = \int t^3\cos t\,dt$$

$t = \arcsin x$

$x = \sin t \Rightarrow dx = \cos t\,dt$

TA PRIMER REŠIMO Z TRIKRATNO INTEGRACIJO PO DELIH. VSAKIČ ZA u VZAMEMO POLINOM.

REZULTAT:

$$\int \arcsin^3 x\,dx = t^3\sin t + 3t^2\cos t - 6t\sin t - 6\cos t + C$$

ČE UPORABIMO $\sin(\arcsin x) = x$ IN $\cos(\arcsin x) = \sqrt{1-x^2}$, DOBIMO:

$$\int \arcsin^3 x\,dx = x\arcsin^3 x + 3\sqrt{1-x^2}\arcsin^2 x - 6x\arcsin x - 6\sqrt{1-x^2} + C$$

④ (a) $I = \int x^2 e^{3x}\,dx = \frac{x^2}{3}e^{3x} - \frac{2}{3}\int x e^{3x}\,dx =$

$u = x^2 \qquad dv = e^{3x}\,dx$

$du = 2x\,dx \qquad v = \frac{1}{3}e^{3x}$

$u = x \qquad dv = e^{3x}\,dx$

$du = dx \qquad v = \frac{1}{3}e^{3x}$

$$I = \frac{x^2}{3}e^{3x} - \frac{2}{9}x e^{3x} + \frac{2}{9}\int e^{3x}\,dx$$

$$= \left(\frac{x^2}{3} - \frac{2}{9}x + \frac{2}{27}\right)e^{3x} + C$$

(b) $I = \int \ln^2 x\,dx = x\ln^2 x - 2\int \ln x\,dx$

$u = \ln^2 x \qquad dv = dx$

$du = 2\cdot\ln x\cdot\frac{1}{x}\,dx \qquad v = x$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}\,dx \qquad v = x$

$$I = x\ln^2 x - 2x\ln x + \int dx = x\left(\ln^2 x - 2\ln x + 1\right) + C$$

(c) $I = \int \frac{\ln(\ln x)}{x}\,dx = \int \ln(t)\,dt = t\ln(t) - t + C =$

$t = \ln x$

$dt = \frac{dx}{x}$

ISTI PER PARTES KOT ZGORAJ

$$= \ln(x)\cdot\ln(\ln x) - \ln(x) + C$$