24.3.2021

KRIVULJE

Parametrizacija krivulje

$\vec{r} : [\alpha, \beta] \to \mathbb{R}^2$ (ali $\mathbb{R}^3$)

za $\forall t \in [\alpha, \beta]$ velja $\|\dot{\vec{r}}(t)\| \neq 0$ (niso vsi odvodi koordinatnih funkcij hkrati enaki 0)

Ista krivulja ima lahko več različnih parametrizacij (pomembna je naravna parametrizacija).

Dolžina krivulje: $l = \int_\alpha^\beta \|\dot{\vec{r}}(t)\| dt$

Parametrizacija z naravnim parametrom:

$$s(t) = \int_\alpha^t \sqrt{(\dot{x}(u))^2 + (\dot{y}(u))^2 + (\dot{z}(u))^2}\, du$$

PODAJANJE KRIVULJ

- PARAMETRIČNO: $I \subseteq \mathbb{R}$ interval $\vec{r} : I \to \mathbb{R}^2$ ali $\mathbb{R}^3$
- EKSPLICITNO: $y = f(x)$ (v ravnini)
- IMPLICITNO: $F(x,y) = 0$ (v $\mathbb{R}^2$)

  oz. $\left.\begin{array}{l} F(x,y,z) = 0 \\ G(x,y,z) = 0 \end{array}\right\}$ (v $\mathbb{R}^3$)

RISANJE KRIVULJ

$\vec{r}$ ... parametrizacija krivulje

$\vec{r}(t)$ si predstavljamo kot položaj točke ob času t

$\dot{\vec{r}}(t)$ ... hitrost točke v času t

Pri risanju si pomagamo z:

- grafi in stacionarnimi točkami komponent
- limitami (opazujemo asimptotično obnašanje komponent)

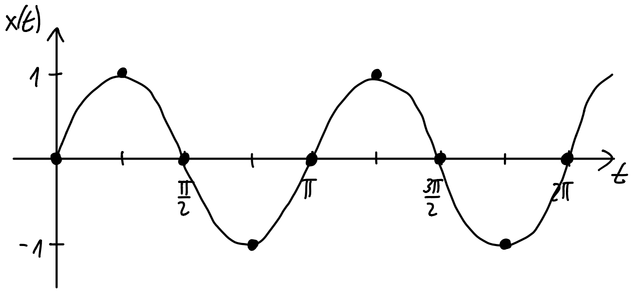
Tangenta na krivuljo: $k = \frac{\dot{y}(t)}{\dot{x}(t)}$

$y = k \cdot x + n$

N: Nariši krivuljo $\vec{r}(t) = (\sin(2t), \cos(t))$.

Perioda: 2π (tir krivulje na $[0, 2\pi]$ je isti kot na $[k\cdot 2\pi, (k+1)\cdot 2\pi]$)

$x(t) = \sin 2t$
$\dot{x}(t) = 2\cdot\cos 2t$

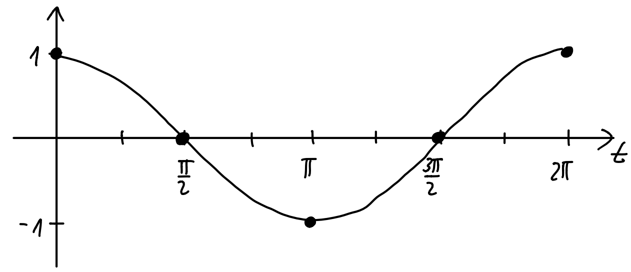


$x(t) = 0: \; t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

$\dot{x}(t) = 0: \; t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$y(t) = \cos t$

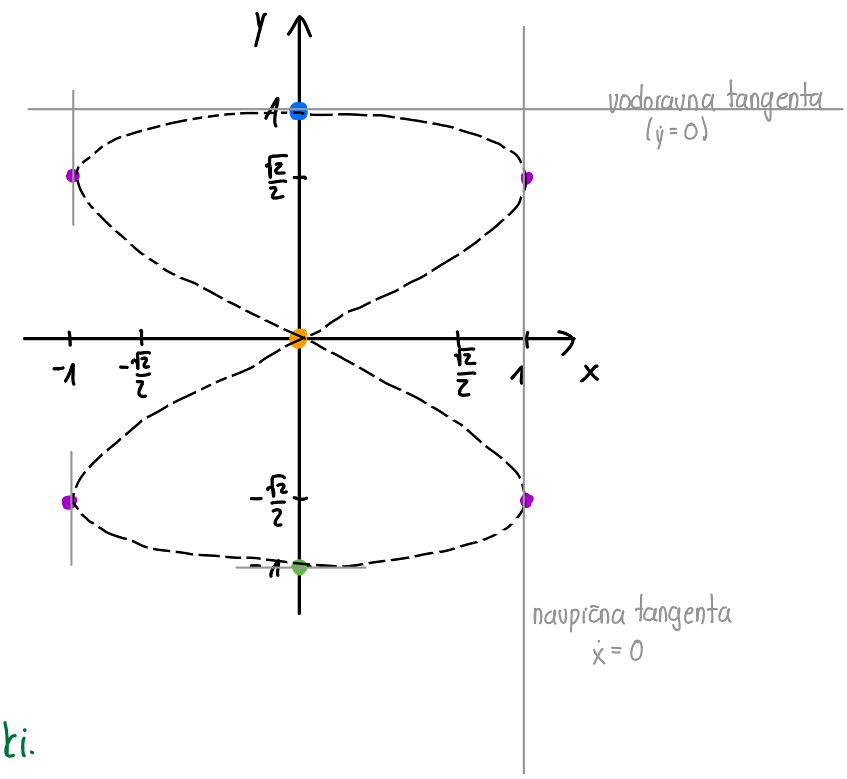
$\dot{y}(t) = -\sin t$



$y(t) = 0: \; t = \frac{\pi}{2}, \frac{3\pi}{2}$

$\dot{y}(t) = 0: \; t = 0, \pi, 2\pi$

t	x	y
0	0	1
$\frac{\pi}{4}$	1	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{2}$	0	0
$\frac{3\pi}{4}$	-1	$-\frac{\sqrt{2}}{2}$
π	0	-1
$\frac{5\pi}{4}$	1	$-\frac{\sqrt{2}}{2}$
$\frac{3\pi}{2}$	0	0
$\frac{7\pi}{4}$	-1	$\frac{\sqrt{2}}{2}$
2π	0	1



Zapiši enačbo krivulje v implicitni obliki.

$\sin^2(2t) = 4\cdot\sin^2 t\cdot\cos^2 t$
$4\cdot\cos^2 t - 4\cdot\sin^2 t\cdot\cos^2 t = 4\cdot\cos^2 t\cdot(1-\sin^2 t) = 4\cdot\cos^4 t$
$4\cdot\cos^2 t - \sin^2(2t) = 4\cdot\cos^4 t$

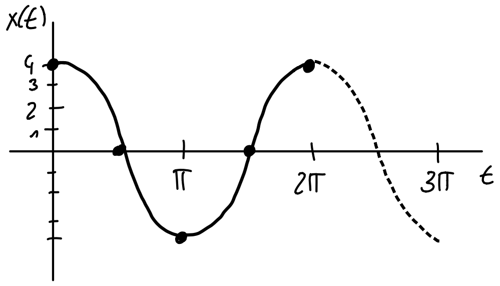
$4y^2 - x^2 = 4x^4$ oz. $4y^2(y^2-1)+x^2 = 0$

1: Nariši krivuljo, ki je podana s parametrizacijo $x(t) = 4\cdot\cos t$ in $y(t) = 3\cdot\sin t$.
Enačbo krivulje zapiši tudi v implicitni obliki

Perioda 2π

$x(t) = 4\cdot\cos t$

$\dot{x}(t) = -4\cdot\sin t$

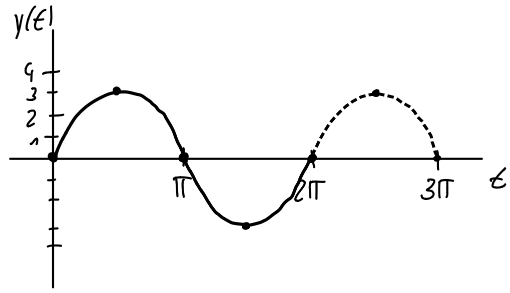


$x(t) = 0: \; t = \frac{\pi}{2}, \frac{3\pi}{2}$

$\dot{x}(t) = 0: \; t = 0, \pi, 2\pi$

$y(t) = 3\cdot\sin t$

$\dot{y}(t) = 3\cdot\cos t$

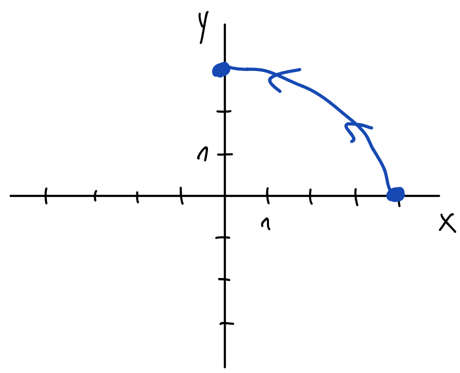


$y(t) = 0: \; t = 0, \pi, 2\pi$

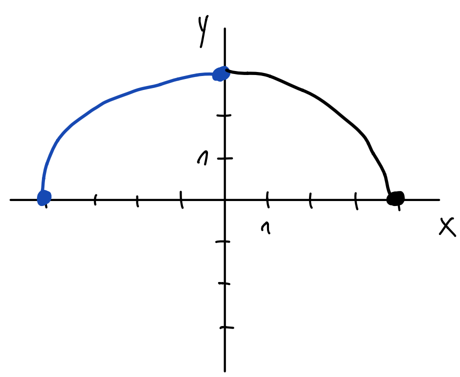
$\dot{y}(t) = 0: \; t = \frac{\pi}{2}, \frac{3\pi}{2}$

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$x(t)$	4	0	-4	0	4
$y(t)$	0	3	0	-3	0

$t \in [0, \frac{\pi}{2}]$: potujemo od točke $(4,0)$ do točke $(0,3)$
v levo (x pada) in navzgor (y narašča)

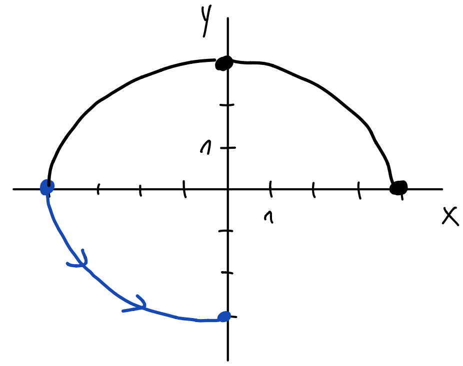


$t \in [\frac{\pi}{2}, \pi]$: potujemo od točke $(0,3)$ do točke $(-4,0)$
v levo (x pada) in navzdol (y pada)



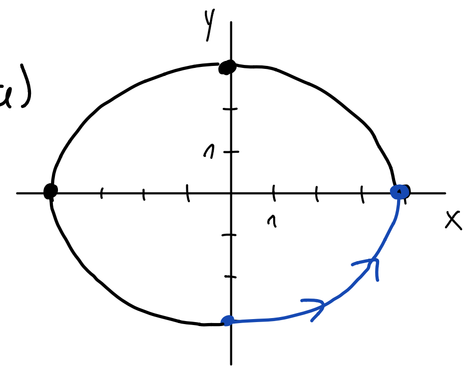
$t \in [\pi, \frac{3\pi}{2}]$: potujemo od točke $(-4,0)$ do točke $(0,-3)$

v desno (x narašča) in navzdol (y pada)



$t \in [\frac{3\pi}{2}, 2\pi]$: potujemo od točke $(0,-3)$ do točke $(4,0)$

v desno (x narašča) in navzgor (y narašča)



Implicitna oblika:

$$x = 4 \cdot \cos t \quad \Rightarrow \quad t = \arccos \frac{x}{4}$$

$$y = 3 \cdot \sin t$$

$$\frac{y}{3} = \sin t \quad /^2$$

$$\frac{y^2}{3^2} = \sin^2 t = 1 - \cos^2 t = 1 - \cos^2\left(\arccos \frac{x}{4}\right) = 1 - \left(\frac{x}{4}\right)^2$$

$$\frac{y^2}{3^2} = 1 - \left(\frac{x}{4}\right)^2$$

$$\frac{y^2}{3^2} + \frac{x^2}{4^2} = 1$$ elipsa z $a = 4$ in $b = 3$

10: Nariši Descartesov list, t.j. krivuljo, ki je podana v implicitni obliki z enačbo

$$x^3-3xy+y^3=0.$$

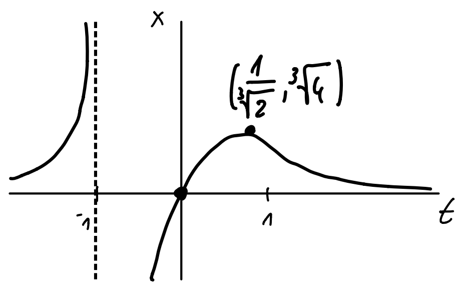
Nasvet: uvedi parameter $t=\frac{y}{x}$.

$$x^3-3xy+y^3=0 \quad /:x^3$$
$$1-3\frac{y}{x}\cdot\frac{1}{x}+\left(\frac{y}{x}\right)^3=0$$
$$1-3t\cdot\frac{1}{x}+t^3=0$$
$$3t\cdot\frac{1}{x}=1+t^3$$
$$x=\frac{3t}{1+t^3}$$

$$y=t\cdot x$$
$$y=\frac{3t^2}{1+t^3}$$

$$x(t)=\frac{3t}{1+t^3}$$

$$\dot{x}(t)=\frac{3(1+t^3)-3t^2\cdot 3t}{(1+t^3)^2}=$$
$$=\frac{3-6t^3}{(1+t^3)^2}$$

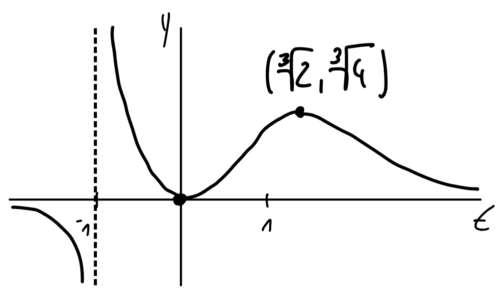


$x(t)=0$: $t=0$

$\dot{x}(t)=0$: $1-2t^3=0$
$$t^3=\frac{1}{2}$$
$$t=\frac{1}{\sqrt[3]{2}}\approx 0{,}8$$
$$x\left(\frac{1}{\sqrt[3]{2}}\right)=\sqrt[3]{4}$$
$$y\left(\frac{1}{\sqrt[3]{2}}\right)=\sqrt[3]{2}$$

$$y(t)=\frac{3t^2}{1+t^3}$$

$$\dot{y}(t)=\frac{6t(1+t^3)-3t^2\cdot 3t^2}{(1+t^3)^2}=$$
$$=\frac{6t-3t^4}{(1+t^3)^2}$$



$y(t)=0$: $t_{1,2}=0$

$\dot{y}(t)=0$: $t=0$
$$6-3t^3=0$$
$$2-t^3=0$$
$$t^3=2$$
$$t=\sqrt[3]{2}\approx 1{,}3$$
$$y(\sqrt[3]{2})=\sqrt[3]{4}$$
$$x(\sqrt[3]{2})=\sqrt[3]{2}$$

$t=0$: $\quad \vec{r}(0) = (0,0)$ V tej točki je tangenta na krivuljo vodoravna, ker je $\dot{y}(0)=0$

$t \in (0, \frac{1}{\sqrt[3]{2}})$ potujemo desno (x narašča) in navzgor (y narašča)

$t = \frac{1}{\sqrt[3]{2}}$ $\quad \vec{r}(\frac{1}{\sqrt[3]{2}}) = (\sqrt[3]{4}, \sqrt[3]{2})$ navpična tangenta $(\dot{x}(\frac{1}{\sqrt[3]{2}}) = 0)$

$t \in (\frac{1}{\sqrt[3]{2}}, \sqrt[3]{2})$ potujemo levo in navzgor

$t = \sqrt[3]{2}$ $\quad \vec{r}(\sqrt[3]{2}) = (\sqrt[3]{2}, \sqrt[3]{4})$ vodoravna tangenta $(\dot{y}(\sqrt[3]{2}) = 0)$

$t \in (\sqrt[3]{2}, \infty)$ potujemo levo in navzdol

$t \to \infty$: $\quad \lim\limits_{t\to\infty} x(t) = 0, \ \lim\limits_{t\to\infty} y(t) = 0 \Rightarrow \lim\limits_{t\to\infty} r(t) = (0,0)$

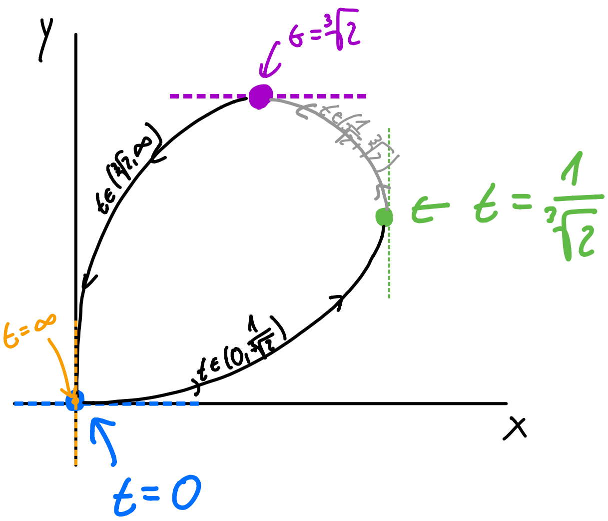
naklon tangente v $(0,0)$:

naklon tangente na krivuljo v položaju, ki ustreza parametru t, je enak

$$k(t) = \frac{\dot{y}}{\dot{x}}$$

V našem primeru:
$$k(t) = \frac{\frac{3at(2-t^3)}{(1+t^3)^2}}{\frac{3a(1-2t^3)}{(1+t^3)^2}} = \frac{t(2-t^3)}{1-2t^3}$$

$$\lim_{t\to\infty} k(t) = \lim_{t\to\infty} \frac{-t^4+2t}{-2t^3+1} = \infty \Rightarrow \text{navpična tangenta}$$



Mi smo narisali del krivulje za $t \in [0,\infty)$ (omejen del)

DODATNO (nismo naredili na vajah): dodamo preostali del krivulje ($t<0$)

$t\in(-1,0)$: če t pada proti -1: $\lim\limits_{t\downarrow -1} x(t) = -\infty$ in $\lim\limits_{t\downarrow -1} y(t) = \infty$

ko gremo s t od 0 do -1, potujemo od točke $(0,0)$ v neskončnost levo in navzgor

$$\lim_{t\downarrow -1} k(t) = \lim_{t\downarrow -1} \frac{-t^4+2t}{-2t^3+1} = \frac{-3}{3} = -1$$

$$y=k\cdot x+n:\ \lim_{t\downarrow -1} n(t) = \lim_{t\downarrow -1} \left(y(t)-k(t)\cdot x(t)\right) = \lim_{t\downarrow -1} \left(y(t)+x(t)\right) = \lim_{t\downarrow -1} \frac{3t+3t^2}{1+t^3} =$$

$$= \lim_{t\downarrow -1} \frac{3\cdot t\cdot \cancel{(1+t)}}{\cancel{(1+t)}(1-t+t^2)} = \frac{-3}{3} = -1$$

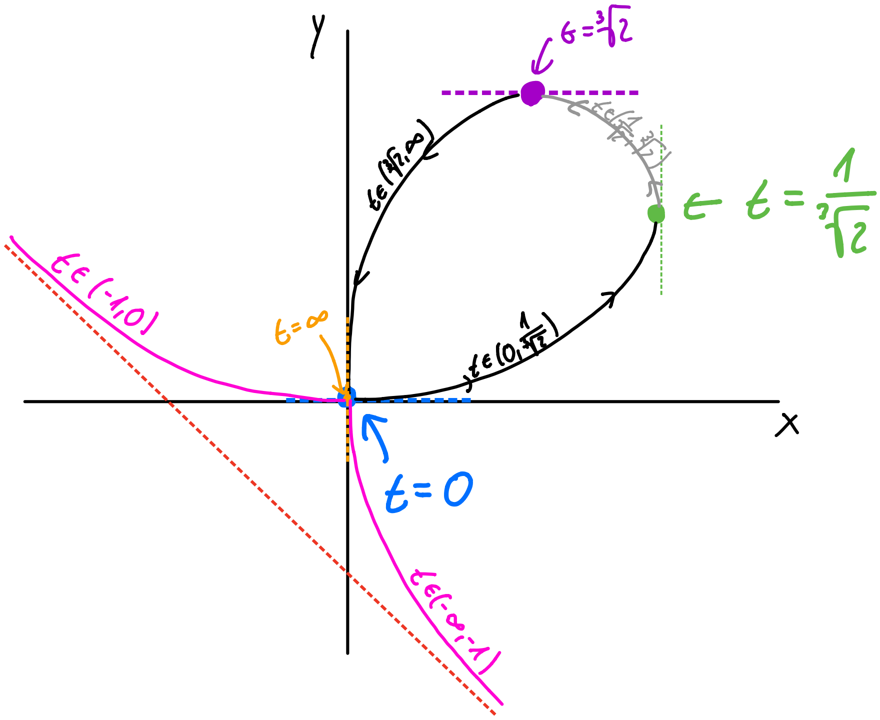
$\Rightarrow$ približujemo se poševni asimptoti $y=-x-1$

$t\in(-\infty,-1)$: če t narašča proti -1: $\lim\limits_{t\uparrow -1} x(t) = \infty$ $\lim\limits_{t\uparrow -1} y(t) = -\infty$

$$\lim_{t\to -\infty} r(t) = (0,0)$$

Ko s t potujemo od $-\infty$ do -1, potujemo od točke $(0,0)$ proti neskončnosti v desno in navzdol

Podobno kot prej pokažemo, da se približujemo poševni asimptoti $y=-x-1$.



(N:) Parametriziraj daljico od točke $A(-1,2,0)$ do točke $B(-7,2,8)$ z naravnim parametrom.

Daljica: $\vec{r}(t) = (-1,2,0) + t\cdot\left((-7,2,8) - (-1,2,0)\right) =$ $\qquad t \in [0,1]$

$= (-1-6t, 2, 8t)$

$$s(t) = \int_0^t |\dot{\vec{r}}(u)|\,du = \int_0^t |(-6,0,8)|\,du = \int_0^t \sqrt{(-6)^2+0^2+8^2}\,du = \int_0^t \sqrt{100}\,du = 10t$$

$s(0)=0,\ s(1)=10 \Rightarrow s\in[0,10]$

$s = 10t \Rightarrow t = \frac{s}{10}$

$\vec{r}(s) = \left(-1-6\frac{s}{10}, 2, 8\frac{s}{10}\right) = \left(-1-\frac{3s}{5}, 2, \frac{4s}{5}\right)$

(N:) Izračunaj dolžino polovice loka cikloide z radijem 1, t.j.

$x(\varphi) = \varphi - \sin\varphi$

$y(\varphi) = 1-\cos\varphi$ $\qquad \varphi \in [0,\pi]$

$r(\varphi) = (\varphi-\sin\varphi, 1-\cos\varphi)$

$\dot{r}(\varphi) = (1-\cos\varphi, \sin\varphi)$

$$|\dot{r}(\varphi)| = \sqrt{(1-\cos\varphi)^2+\sin^2\varphi} = \sqrt{1-2\cos\varphi+1} = \sqrt{2-2\cos\varphi} = \sqrt{2}\cdot\sqrt{1-\cos\varphi} =$$

$$= \sqrt{2}\cdot\sqrt{1-\left(\cos^2\frac{\varphi}{2}-\sin^2\frac{\varphi}{2}\right)} = \sqrt{2}\cdot\sqrt{2\cdot\sin^2\frac{\varphi}{2}} = 2\cdot\sqrt{\sin^2\frac{\varphi}{2}} = 2\cdot\left|\sin\frac{\varphi}{2}\right|$$

$$l = \int_0^\pi 2\cdot\left|\sin\frac{\varphi}{2}\right|d\varphi = \int_0^{\frac{\pi}{2}} 2\cdot|\sin u|\cdot 2\cdot du = 4\cdot\int_0^{\frac{\pi}{2}}|\sin u|\,du = 4\cdot\int_0^{\frac{\pi}{2}}\sin u\,du =$$

$u = \frac{\varphi}{2}$

$du = \frac{1}{2}d\varphi$

$$= 4\cdot(-\cos u)\Big|_0^{\frac{\pi}{2}} = 4\cdot(-0-(-1)) = \underline{\underline{4}}$$

N: Dana je krivulja $\vec{r}(t) = (3t^2-1,\ t(3-t^2))$. TSCHIRNHAUSENOVA KRIVULJA

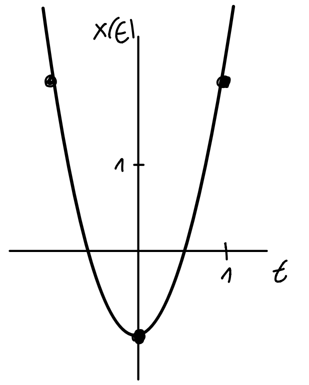
a) skiciraj krivuljo

b) izrazi naravni parameter krivulje z začetnim

c) izračunaj dolžino krivulje med točkama $(-1,0)$ in $(8,0)$

a) $x(t) = 3t^2-1$

$\dot{x}(t) = 6t$



$x(t)=0:\quad t = \pm\frac{1}{\sqrt{3}}$

$y(\pm\frac{1}{\sqrt{3}}) = \pm\frac{1}{\sqrt{3}}\cdot\frac{8}{3}$

$\dot{x}(t)=0:\quad t=0$

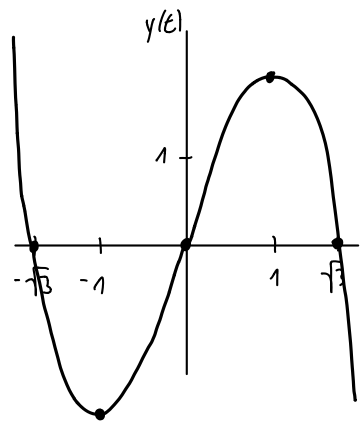
$x(0)=-1$

$y(0)=0$

$y(t) = t\cdot(3-t^2)$

$\dot{y}(t) = 3-t^2+t\cdot(-2t) =$

$= 3-3t^2$



$y(t)=0:\quad t=0,\ t=\pm\sqrt{3}$

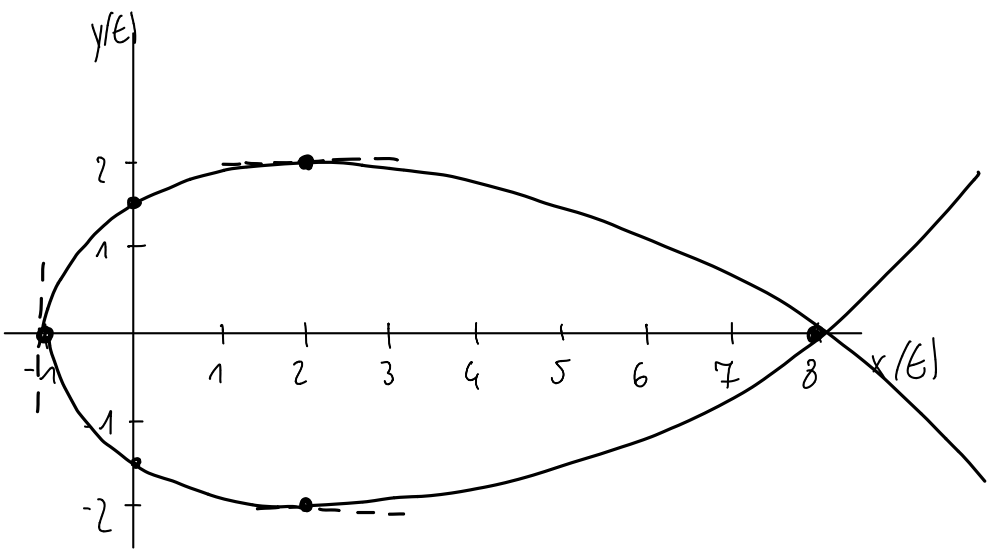
$\dot{y}(t)=0:\quad t=\pm 1$

$x(\pm 1) = 2$

$y(\pm 1) = \pm 2$

$x(\pm\sqrt{3}) = 8$

$y(\pm\sqrt{3}) = 0$



	x	y
	pada	pada
$t=-1$:	2	-2
	pada	narašča
$t=0$:	-1	0
	narašča	narašča
$t=1$:	2	2
	narašča	pada

$$k(t) = \frac{\dot{y}(t)}{\dot{x}(t)} = \frac{3(1-t^2)}{6t} = \frac{1-t^2}{2t}$$

$t \to \infty: \quad k(t) \to -\infty$

$t \to -\infty: \quad k(t) \to \infty$

b) $$s(t) = \int_0^t |\dot{r}(u)|\,du = \int_0^t |(6u, 3-3u^2)|\,du = 3\int_0^t |(2u, 1-u^2)|\,du =$$

ni pomembno
v kateri točki
začnemo

$$= 3\int_0^t \sqrt{4u^2 + (1-u^2)^2}\,du = 3\int_0^t \sqrt{(1+u^2)^2}\,du = 3\int_0^t 1+u^2\,du = 3\cdot\left(u+\frac{u^3}{3}\right)\Big|_0^t =$$

$$= 3t + t^3$$

c) $(-1,0) \rightarrow t = 0$

$3t^2-1=-1$ $\quad t\cdot(3-t^2)=0$
$3t^2=0$
$t^2=0$
$t=0$

$(8,0) \rightarrow t = \pm\sqrt{3}$

$3t^2-1=8$ $\quad t(3-t^2)=0$
$3t^2=9$
$t^2=3$

$$\ell = \int_0^{\sqrt{3}} |\dot{r}(u)|\,du = s(\sqrt{3}) = \underline{\underline{6\sqrt{3}}}$$

ker smo pri b) primeru za
začetno točko izbrali
$t=0$

Enačba tangente na krivuljo $\vec{r}=\vec{r}(t)$ v točki t_0:

$$\vec{T}(\lambda) = \vec{r}(t_0) + \lambda \cdot \dot{\vec{r}}(t_0)$$

Normalna ravnina na krivuljo $\vec{r}=\vec{r}(t)$ v točki t_0: ravnina, ki vsebuje $\vec{r}(t_0)$ z normalo $\dot{\vec{r}}(t_0)$.

N: Izračunaj enačbo tangente in normalne ravnine na krivuljo s parametrizacijo $\vec{r}(t) = (t, \frac{t^2}{2}, -t^3)$ v točki $(1, \frac{1}{2}, -1)$

$$\dot{\vec{r}}(t) = (1, t, -3t^2)$$

$$\vec{r}(t_0) = (1, \tfrac{1}{2}, -1) \Rightarrow t_0 = 1$$

$$\vec{T}(\lambda) = (1, \tfrac{1}{2}, -1) + \lambda \cdot \dot{\vec{r}}(1) = (1, \tfrac{1}{2}, -1) + \lambda \cdot (1, 1, -3)$$

Normalna ravnina:

$$\langle \underbrace{(x, y, z)}_{\text{točka na normalni ravnini}} - (1, \tfrac{1}{2}, -1), (1, 1, -3) \rangle = 0$$

$$(x-1)\cdot 1 + (y-\tfrac{1}{2})\cdot 1 + (z+1)\cdot(-3) = 0$$

$$x - 1 + y - \tfrac{1}{2} - 3z - 3 = 0$$

$$x + y - 3z = \tfrac{9}{2}$$ ← enačba normalne ravnine

D: Parametriziraj krivuljo $x^2+y^2+z^2=4$, $(x-1)^2+y^2=1$ in zapiši enačbo tangente pri $x=1$, $y<0$, $z>0$.

Cilindrične koordinate:

$x = r\cdot\cos\varphi$

$y = r\cdot\sin\varphi$

$z = z$

$x^2+y^2+z^2=4$

$r^2\cdot\cos^2\varphi + r^2\cdot\sin^2\varphi + z^2 = 4$

$r^2+z^2=4$

$(x-1)^2+y^2=1$

$r^2\cdot\cos^2\varphi - 2\cdot\cos\varphi + \cancel{1} + r^2\cdot\sin^2\varphi = \cancel{1}$

$r^2 - 2\cdot r\cdot\cos\varphi = 0 \;/ : r$

$r = 2\cdot\cos\varphi$

$z^2 = 4-r^2 = 4-4\cos^2\varphi = 4\sin^2\varphi$

$z = \pm 2\sin\varphi$

$x = 2\cdot\cos^2\varphi$

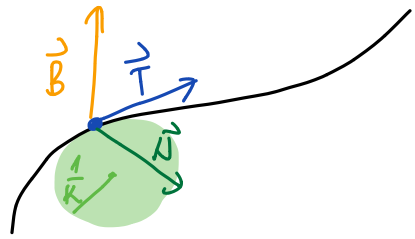
$y = 2\cdot\cos\varphi\cdot\sin\varphi \qquad \varphi\in[0,2\pi]$

$z = 2\cdot\sin\varphi$

TANGENTA: DN

UKRIVLJENOST

SPREMLJAJOČI TRIEDER krivulje s parametrizacijo $\vec{r}(t)$ v točki $\vec{r}(t_0)$ sestavljajo:



- TANGENTNI VEKTOR $\vec{T} = \dfrac{\dot{\vec{r}}(t_0)}{\|\dot{\vec{r}}(t_0)\|} = \vec{r}\,'(s_0)$
- VEKTOR GLAVNE NORMALE $\vec{N} = \vec{B} \times \vec{T} = \dfrac{\vec{r}\,''(s_0)}{\|\vec{r}\,''(s_0)\|} = \dfrac{\|\dot{\vec{r}}(t_0)\|^2 \ddot{\vec{r}}(t_0) - (\dot{\vec{r}}(t_0) \cdot \ddot{\vec{r}}(t_0)) \cdot \dot{\vec{r}}(t_0))}{\left\| \|\dot{\vec{r}}(t_0)\|^2 \ddot{\vec{r}}(t_0) - (\dot{\vec{r}}(t_0) \cdot \ddot{\vec{r}}(t_0)) \cdot \dot{\vec{r}}(t_0)) \right\|}$
- VEKTOR BINORMALE $\vec{B} = \dfrac{\dot{\vec{r}}(t_0) \times \ddot{\vec{r}}(t_0)}{\|\dot{\vec{r}}(t_0) \times \ddot{\vec{r}}(t_0)\|}$

PRITISNJENA RAVNINA krivulje s parametrizacijo $\vec{r} = \vec{r}(t)$ v točki $\vec{r}(t_0)$ ima normalo $\vec{B}$

NORMALNA RAVNINA krivulje s parametrizacijo $\vec{r} = \vec{r}(t)$ v točki $\vec{r}(t_0)$ ima normalo $\vec{T}$

UPOGNJENOST (FLEKSIJSKA UKRIVLJENOST) krivulje s parametrizacijo $\vec{r} = \vec{r}(t)$ v točki $\vec{r}(t_0)$

je enaka

$$\kappa = \frac{\|\dot{\vec{r}}(t_0) \times \ddot{\vec{r}}(t_0)\|}{\|\dot{\vec{r}}(t_0)\|^3} = \|\vec{r}\,''(s_0)\|$$

POLMER PRITISNJENE KROŽNICE: $\frac{1}{\kappa}$

ZVITOST (TORZIJSKA UKRIVLJENOST) krivulje s parametrizacijo $\vec{r} = \vec{r}(t)$ v točki $\vec{r}(t_0)$

je enaka

$$\omega = \frac{[\dot{\vec{r}}(t_0), \ddot{\vec{r}}(t_0), \dddot{\vec{r}}(t_0)]}{\|\dot{\vec{r}}(t_0) \times \ddot{\vec{r}}(t_0)\|^2} = \frac{(\dot{\vec{r}}(t_0) \times \ddot{\vec{r}}(t_0)) \cdot \dddot{\vec{r}}(t_0)}{\|\dot{\vec{r}}(t_0) \times \ddot{\vec{r}}(t_0)\|^2} = \|\vec{B}\,'(s_0)\|$$

FRENETOVE FORMULE (v naravni parametrizaciji):

$$\vec{T}\,' = \kappa \cdot \vec{N}$$
$$\vec{B}\,' = -\omega \cdot \vec{N}$$
$$\vec{N}\,' = \omega \cdot \vec{B} - \kappa \cdot \vec{T}$$

* če je $\omega = 0$ za $\forall t$, je krivulja ravninska

1: Krivulja K je določena kot presek ploskev z enačbama $x^2 = 3y$ in $2xy = 9z$.

a) Parametriziraj krivuljo

b) Določi tangentni vektor v točki $(3,3,2)$.

c) Določi naravni parameter za krivuljo K in dolžino odseka krivulje K med točkama $(0,0,0)$ in $(3,3,2)$.

d) Izračunaj upognjenost (fleksijsko ukrivljenost) in zvitost (torzijsko ukrivljenost) krivulje K v točki $(0,0,0)$.

a) $x^2 = 3y \Rightarrow y = \frac{x^2}{3}$

$$2xy = 9z \Rightarrow z = \frac{2}{9}xy = \frac{2}{27}x^3$$

$$\vec{r}(x) = \left(x, \frac{x^2}{3}, \frac{2}{27}x^3\right)$$

$$\vec{r}(t) = \left(t, \frac{1}{3}t^2, \frac{2}{27}t^3\right)$$

b) $(3,3,2) = \vec{r}(t_0)$

$$\left.\begin{aligned} 3 &= t_0 \\ 3 &= \tfrac{1}{3}t_0^2 \\ 2 &= \tfrac{2}{9}t_0^2 \end{aligned}\right\} t_0 = 3$$

$$\dot{\vec{r}}(t) = \left(1, \frac{2}{3}t, \frac{2\cdot 3}{27}t^2\right) = \left(1, \frac{2}{3}t, \frac{2}{9}t^2\right)$$

$$\dot{\vec{r}}(3) = (1,2,2)$$

$$\vec{T}(3) = \frac{(1,2,2)}{\|(1,2,2)\|} = \frac{(1,2,2)}{\sqrt{1^2+2^2+2^2}} = \frac{1}{3}(1,2,2)$$

c) $$s(t) = \int_0^t |\dot{\vec{r}}(u)|\,du = \int_0^t \sqrt{1 + \frac{4u^2}{9} + \frac{4u^4}{81}}\,du = \int_0^t \sqrt{\left(1+\frac{2u^2}{9}\right)^2}\,du = \int_0^t \left(1+\frac{2u^2}{9}\right)du = t + \frac{2t^3}{27}$$

$(3,3,2) \rightarrow t = 3$
$(0,0,0) \rightarrow t = 0$

$$l = s(3) - s(0) = 3+2-0 = 5$$

d) $\dot{\vec{r}}(t) = (1, \frac{2}{3}t, \frac{2}{9}t^2)$ $\qquad$ $\dot{\vec{r}}(0) = (1,0,0)$

$\ddot{\vec{r}}(t) = (0, \frac{2}{3}, \frac{4}{9}t)$ $\qquad$ $\ddot{\vec{r}}(0) = (0, \frac{2}{3}, 0)$

$\dddot{\vec{r}}(t) = (0, 0, \frac{4}{9})$ $\qquad$ $\dddot{\vec{r}}(0) = (0, 0, \frac{4}{9})$

$$\dot{\vec{r}}(0) \times \ddot{\vec{r}}(0) = (1,0,0) \times (0, \tfrac{2}{3}, 0) = \begin{array}{ccccc} x & y & z & x & y \\ 1 & 0 & 0 & 1 & 0 \\ 0 & \frac{2}{3} & 0 & 0 & \frac{2}{3} \end{array} =$$

$$= (0-0, 0-0, \tfrac{2}{3}-0) = (0, 0, \tfrac{2}{3})$$

$$\kappa = \frac{\|(0,0,\frac{2}{3})\|}{\|(1,0,0)\|^3} = \frac{2}{3}$$

$$w = \frac{(0,0,\frac{2}{3}) \cdot (0,0,\frac{4}{9})}{\|(0,0,\frac{2}{3})\|^2} = \frac{\frac{8}{27}}{\frac{4}{9}} = \frac{2}{3}$$

N: Naj bo $\vec{r}(t) = (e^t\cos t, e^t\sin t, e^t)$.

a) Določi naravni parameter.

b) Izračunaj spremljajoči trieder.

c) Izračunaj fleksijsko in torzijsko ukrivljenost krivulje v točki $(1,0,1)$.

d) Določi enačbo pritisnjene ravnine v točki $(1,0,1)$.

$$\dot{\vec{r}}(t) = (e^t\cdot\cos t - e^t\cdot\sin t,\ e^t\cdot\sin t + e^t\cdot\cos t,\ e^t)$$

$$\ddot{\vec{r}}(t) = (e^t\cdot(\cancel{\cos t} - \sin t - \sin t - \cancel{\cos t}),\ e^t(\cancel{\sin t} + \cos t + \cos t - \cancel{\sin t}),\ e^t) =$$

$$= (-2\cdot e^t\cdot\sin t,\ 2\cdot e^t\cdot\cos t,\ e^t)$$

$$\dddot{\vec{r}}(t) = (-2\cdot e^t\cdot(\sin t + \cos t),\ 2\cdot e^t(\cos t - \sin t),\ e^t)$$

a) $$\|\dot{\vec{r}}(t)\| = \sqrt{(e^t\cdot(\cos t - \sin t))^2 + (e^t\cdot(\sin t + \cos t))^2 + (e^t)^2} =$$

$$= \sqrt{e^{2t}\cdot(\cos^2 t - \cancel{2\sin t\cos t} + \sin^2 t + \sin^2 t + \cancel{2\sin t\cos t} + \cos^2 t + 1)} = e^t\cdot\sqrt{3}$$

$$s(t) = \int_0^t \|\dot{\vec{r}}(u)\|\,du = \int_0^t \sqrt{3}\cdot e^u\,du = \sqrt{3}e^u\Big|_0^t = \sqrt{3}(e^t - 1)$$

b) $$\vec{T}(t) = \frac{\dot{\vec{r}}(t)}{\|\dot{\vec{r}}(t)\|} = \frac{1}{\sqrt{3}e^t}\left(e^t(\cos t - \sin t),\ e^t(\sin t + \cos t),\ e^t\right) =$$

$$= \frac{1}{\sqrt{3}}(\cos t - \sin t,\ \sin t + \cos t,\ 1)$$

$$\dot{\vec{r}}(t) \times \ddot{\vec{r}}(t) = \begin{array}{ccc} x & y & z \\ e^t(\cos t - \sin t) & e^t(\sin t + \cos t) & e^t \\ -2e^t\sin t & 2e^t\cos t & e^t \end{array} =$$

$$= (e^{2t}(\sin t + \cos t) - 2e^{2t}\cos t,\ -2e^{2t}\sin t - e^{2t}(\cos t - \sin t),\ 2e^{2t}\cos t(\cos t - \sin t) + 2e^{2t}\sin t(\sin t + \cos t)) =$$

$$= (e^{2t}(\sin t - \cos t),\ e^{2t}(-\sin t - \cos t),\ 2e^{2t}(\cos^2 t - \cancel{\cos t\sin t} + \sin^2 t + \cancel{\sin t\cos t})) =$$

$$= e^{2t}(\sin t - \cos t,\ -\sin t - \cos t,\ 2)$$

$$\|\dot{\vec{r}}(t) \times \ddot{\vec{r}}(t)\| = e^{2t} \cdot \sqrt{(\sin t - \cos t)^2 + (-\sin t - \cos t)^2 + 2^2} =$$

$$= \sqrt{6} \cdot e^{2t} \sqrt{\sin^2 t - 2\sin t \cancel{\cos t} + \cos^2 t + \sin^2 t + 2\sin \cancel{t} \cos t + \cos^2 t + 4} =$$

$$= \sqrt{6} \cdot e^{2t}$$

$$\vec{B}(t) = \frac{e^{2t}(\sin t - \cos t, -\sin t - \cos t, 2)}{\|e^{2t}(\sin t - \cos t, -\sin t - \cos t, 2)\|} = \frac{1}{\sqrt{6}} \cdot \frac{1}{\cancel{e^{2t}}} \cdot \cancel{e^{2t}}(\sin t - \cos t, -\sin t - \cos t, 2) =$$

$$= \frac{1}{\sqrt{6}}(\sin t - \cos t, -\sin t - \cos t, 2)$$

$$\vec{N}(t) = \vec{B}(t) \times \vec{T}(t) = \frac{1}{\sqrt{6}}(\sin t - \cos t, -\sin t - \cos t, 2) \times \frac{1}{\sqrt{3}}(\cos t - \sin t, \sin t + \cos t, 1) =$$

$$= \frac{1}{3\sqrt{2}} \begin{vmatrix} x & y & z \\ (\sin t - \cos t) & (-\sin t - \cos t) & 2 \\ (\cos t - \sin t) & (\sin t + \cos t) & 1 \end{vmatrix} =$$

$$= \frac{1}{3\sqrt{2}} \Big((-\sin t - \cos t) \cdot 1 - 2 \cdot (\sin t + \cos t), 2 \cdot (\cos t - \sin t) - 1 \cdot (\sin t - \cos t),$$
$$(\sin t - \cos t) \cdot (\sin t + \cos t) - (-\sin t - \cos t) \cdot (\cos t - \sin t) \Big) =$$

$$= \frac{1}{3\sqrt{2}} \left(-3(\sin t + \cos t), 3(\cos t - \sin t), \cancel{\sin^2 t} - \cancel{\cos^2 t} + \cancel{\cos^2 t} - \cancel{\sin^2 t} \right) =$$

$$= \frac{1}{\sqrt{2}}(-\sin t - \cos t, \cos t - \sin t, 0)$$

c) $(1,0,1) = \vec{r}(0)$

$$K = \frac{\|\dot{\vec{r}}(0) \times \ddot{\vec{r}}(0)\|}{\|\dot{\vec{r}}(0)\|^3} = \frac{\sqrt{6} \cdot e^{2 \cdot 0}}{(e^0 \cdot \sqrt{3})^3} = \frac{\sqrt{6}}{\sqrt{3}^3} = \underline{\underline{\frac{\sqrt{2}}{3}}}$$

$$\omega = \frac{[\dot{\vec{r}}(0),\ddot{\vec{r}}(0),\dddot{\vec{r}}(0)]}{\|\dot{\vec{r}}(0)\times\ddot{\vec{r}}(0)\|^2} = \frac{e^{2\cdot 0}\cdot(\sin 0-\cos 0,-\sin 0-\cos 0,2)\cdot e^{0}\cdot(-2\cdot(\sin 0+\cos 0),2\cdot(\cos 0-\sin 0),1)}{(\sqrt{6}\cdot e^{2\cdot 0})^2} =$$

$$= \frac{(-1,-1,2)\cdot(-2,2,1)}{6} = \frac{1}{6}\cdot(2-\cancel{2}+\cancel{2}) = \underline{\underline{\frac{1}{3}}}$$

d) Ravnina z normalo $\vec{B}(0)$, ki gre skozi točko $\vec{r}(0)$.

$$\vec{B}(0) = \frac{1}{\sqrt{6}}\cdot(\sin 0-\cos 0,-\sin 0-\cos 0,2) = \frac{1}{\sqrt{6}}(-1,-1,2)$$

Enačba ravnine :

$$\langle (x,y,z)-(1,0,1), \frac{1}{\sqrt{6}}(-1,-1,2)\rangle = 0$$

$$\frac{1}{\sqrt{6}}(x-1)\cdot(-1) + \frac{1}{\sqrt{6}}\cdot(y-0)\cdot(-1) + \frac{1}{\sqrt{6}}\cdot(z-1)\cdot 2 = 0 \quad /\cdot\sqrt{6}$$

$$-x+1-y+2z-2 = 0$$

$$\underline{\underline{-x-y+2z = 1}}$$

(N:) Dana je krivulja $\vec{r}(t) = (3\cos t, 3\sin t, 4t)$.

a) Parametriziraj po naravnem parametru.

b) Izračunaj spremljajoči trieder

c) Izračunaj fleksijsko in torzijsko ukrivljenost v točki $(0, 3, 2\pi)$

a) $\dot{\vec{r}}(t) = (-3\sin t, 3\cos t, 4)$

$$\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2} = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 16} = \sqrt{9\sin^2 t + 9\cos^2 t + 16} = \sqrt{9+16} =$$

$$= \underline{\underline{5}}$$

$$s(t) = \int_0^t 5\, du = 5u \Big|_0^t = 5t$$

$$\vec{r}(s) = \left(3\cdot\cos\frac{s}{5}, 3\cdot\sin\frac{s}{5}, \frac{4s}{5}\right)$$

b) UPORABIMO FORMULE ZA NARAVNO PARAMETRIZACIJO!

$$\vec{T}(s) = \vec{r}\,'(s) = \left(-\frac{3}{5}\cdot\sin\frac{s}{5}, \frac{3}{5}\cdot\cos\frac{s}{5}, \frac{4}{5}\right)$$

$$\vec{N}(s) = \frac{\vec{r}\,''(s)}{\|\vec{r}\,''(s)\|} = \frac{\left(-\frac{3}{25}\cos\frac{s}{5}, -\frac{3}{25}\sin\frac{s}{5}, 0\right)}{\left\|\left(-\frac{3}{25}\cos\frac{s}{5}, -\frac{3}{25}\sin\frac{s}{5}, 0\right)\right\|} = \frac{25}{3}\cdot\left(-\frac{3}{25}\cos\frac{s}{5}, -\frac{3}{25}\sin\frac{s}{5}, 0\right) =$$

$$= \left(-\cos\frac{s}{5}, -\sin\frac{s}{5}, 0\right)$$

$$\vec{B}(s) = \vec{T}(s) \times \vec{N}(s) = \begin{vmatrix} x & y & z \\ -\frac{3}{5}\sin\frac{s}{5} & \frac{3}{5}\cos\frac{s}{5} & \frac{4}{5} \\ -\cos\frac{s}{5} & -\sin\frac{s}{5} & 0 \end{vmatrix} =$$

$$= \left(0 + \frac{4}{5}\sin\frac{s}{5}, -\frac{4}{5}\cos\frac{s}{5} + 0, \frac{3}{5}\sin^2\frac{s}{5} + \frac{3}{5}\cos^2\frac{s}{5}\right) =$$

$$= \left(\frac{4}{5}\sin\frac{s}{5}, -\frac{4}{5}\cos\frac{s}{5}, \frac{3}{5}\right)$$

c) Točka $(0,3,2\pi)$

$3\cdot\cos\frac{s}{5}=0 \qquad 3\cdot\sin\frac{s}{5}=3 \qquad \frac{4}{5}s=2\pi \quad \Rightarrow \quad s=\frac{5\pi}{2}$

PONOVNO UPOŠTEVAMO NARAVNO PARAMETRIZACIJO!

$\kappa(s_0)=\|\vec{r}''(s_0)\|=\|(-\frac{3}{25}\cos\frac{s}{5},-\frac{3}{25}\sin\frac{s}{5},0)\|$

$\kappa(\frac{5\pi}{2})=\|-\frac{3}{25}\cdot(\cos\frac{5\pi}{2\cdot5},\sin\frac{5\pi}{2\cdot5},0)\|=\|-\frac{3}{25}(\cos\frac{\pi}{2},\sin\frac{\pi}{2},0)\|=\frac{3}{25}\cdot\|(0,1,0)\|=$

$=\underline{\underline{\frac{3}{25}}}$

$\omega(s_0)=\|\vec{B}'(s_0)\|=\|(\frac{4}{25}\cos\frac{s_0}{5},\frac{4}{25}\sin\frac{s_0}{5},0)\|$

$\omega(\frac{5\pi}{2})=\|\frac{4}{25}(\cos\frac{5\pi}{2\cdot5},\sin\frac{5\pi}{2\cdot5},0)\|=\frac{4}{25}\cdot\|(0,1,0)\|=\underline{\underline{\frac{4}{25}}}$

ČE IMAMO KRIVULJO V RAVNINI SE ENAČBE POENOSTAVIJO V:

$$\vec{r}(t) = (x(t), y(t))$$

Okrivljenost: $\kappa = \dfrac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\sqrt{(\dot{x}^2+\dot{y}^2)}^3}$

$\omega = 0$ (RAVNINSKA KRIVULJA)

N: Izračunaj okrivljenost eksplicitno podane krivulje $y = x^4 - x + 1$ v točki $x = 1$.

$x(t) = t \qquad \dot{x}(t) = 1 \qquad \ddot{x}(t) = 0$

$y(t) = t^4 - t + 1 \qquad \dot{y}(t) = 4t^3 - 1 \qquad \ddot{y}(t) = 12t^2$

$$\kappa = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}^3} = \frac{1\cdot 12t^2 - 0\cdot(4t^3-1)}{\sqrt{1^2+(4t^3-1)^2}^3} = \frac{12t^2}{\sqrt{1+(4t^3-1)^2}^3}$$

$$\kappa(1) = \frac{12}{\sqrt{1+(4-1)^2}^3} = \frac{12}{\sqrt{1+3^2}^3} = \frac{12}{\sqrt{10}^3} = \frac{12}{10\sqrt{10}} = \frac{6}{5\sqrt{10}} = \underline{\underline{\frac{3\sqrt{10}}{25}}}$$