Naloga: Raziščite konvergenco Newtonove iteracije.

Rešitev: $f(x)=0$, $x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)}$, $r=0,\ldots$

(1) Naj bo α enostavna ničla f!

$$f(x) = (x-\alpha)\,h(x)\ ;\ h(\alpha)\neq 0.$$

Iteracijska funkcija za Newtonovo iteracijo je $g(x) = x - \frac{f(x)}{f'(x)}$.

$$g(\alpha) = \alpha - \frac{\overbrace{f(\alpha)}^{=0}}{\underbrace{f'(\alpha)}_{\neq 0}} = \alpha \checkmark$$

$$g'(x) = 1 - \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2}$$

$$= \cancel{1} - \cancel{1} + \frac{f(x)f''(x)}{f'(x)^2}$$

$$g'(\alpha) = \frac{\overbrace{f(\alpha)}^{=0}f''(\alpha)}{\underbrace{f'(\alpha)^2}_{\neq 0}} = 0 \Rightarrow \text{red vsaj } 2!!$$

(2) Naj bo α m-kratna ničla, $m \geq 2$.

$$f(x) = (x-\alpha)^m h(x), \quad h(\alpha) \neq 0.$$

$$f'(x) = m(x-\alpha)^{m-1} h(x) + (x-\alpha)^m h'(x)$$

$$f''(x) = m(m-1)(x-\alpha)^{m-2} h(x) + m(x-\alpha)^{m-1} h'(x) + m(x-\alpha)^{m-1} h'(x) + (x-\alpha)^m h''(x)$$

$$g(x) = x - \frac{(x-\alpha)^m h(x)}{m(x-\alpha)^{m-1} h(x) + (x-\alpha)^m h'(x)}$$

$$= x - \frac{(x-\alpha)^{\cancel{m}} h(x)}{\cancel{(x-\alpha)^{m-1}}\left(m h(x) + (x-\alpha) h'(x)\right)}$$

$$\Rightarrow \quad g(\alpha) = \alpha - \frac{(\alpha-\alpha)\cancel{h(\alpha)}}{m\cancel{h(\alpha)} + 0} = \alpha$$

$$g'(x) = \frac{\cancel{(x-\alpha)^m} h(x) \cancel{(x-\alpha)^{m-2}}\left[m(m-1)h(x) + 2m(x-\alpha)h'(x) + (x-\alpha)^2 h''(x)\right]}{\left(\cancel{(x-\alpha)^{m-1}}\left(m h(x) + (x-\alpha)h'(x)\right)\right)^2}$$

$$g'(\alpha) = \frac{\cancel{h(\alpha)}\, \cancel{m}(m-1)\, \cancel{h(\alpha)}}{m^2 \cancel{h(\alpha)^2}} = \frac{m-1}{m}$$

$= 1 - \frac{1}{m} \neq 0$, ker je $m \geq 2$.

$\Rightarrow$ konvergenca je linearna.

Če je ničla enostavna, je red vsaj 2. Kdaj je več?

$$g'(x) = \frac{f(x) f''(x)}{f'(x)^2}$$

$$g''(x) = \frac{\left(f'(x) f''(x) + f(x) f'''(x)\right) f'(x)^2 - f(x) f''(x)\, 2 f'(x) f''(x)}{(f'(x))^4}$$

$$g''(\alpha) = \frac{\cancel{f'(\alpha)^3}\, f''(\alpha)}{(f'(\alpha))^{\cancel{4}}} = \frac{f''(\alpha)}{f'(\alpha)}$$

Red konvergence bo vsaj 3, če je $f''(\alpha)=0$.