PLOSKVE

31.3.2021

PODAJANJE PLOSKEV:

- EKSPLICITNA OBLIKA: ploskev je zapisana kot graf funkcije

$$S = \{(x,y,f(x,y)) : (x,y) \in D\}$$

- IMPLICITNA OBLIKA: $f : \mathbb{R}^3 \to \mathbb{R}$ gladka funkcija. Ploskev:

$$M = \{(x,y,z) : f(x,y,z) = 0\}$$

- PARAMETRIČNA OBLIKA: $D \subseteq \mathbb{R}^2$ odprta množica in $r : D \to \mathbb{R}^3$ preslikava, $r \in C^1$

Ploskev: $r(u,v) = (r_1(u,v), r_2(u,v), r_3(u,v))$

↳ u, v parametra

Parametrizacija je regularna, če vektorja r_u in r_v nista vzporedna ($r_u \times r_v \neq 0$)

$E(u,v) = r_u \cdot r_u = \|r_u\|^2$

$F(u,v) = r_u \cdot r_v$

$G(u,v) = r_v \cdot r_v = \|r_v\|^2$

KOORDINATNE KRIVULJE: če pri parametričnem zapisu fiksiramo enega od parametrov
Kot med koordinatnima krivuljama φ je pri parametričnem zapisu enak kotu med vektorjema $\vec{r}_u$ in $\vec{r}_v$.

$$\cos\varphi = \frac{F}{\sqrt{E \cdot G}}$$

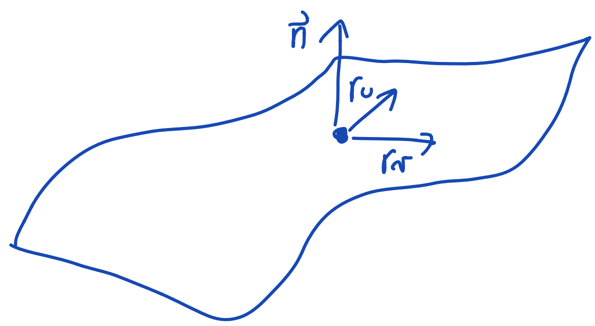
Normalni vektor na ploskev: $\vec{N} = \vec{r}_u \times \vec{r}_v$

Enotski normalni vektor na ploskev: $\vec{n} = \frac{\vec{N}}{\|\vec{N}\|} = \frac{r_u \times r_v}{\|r_u \times r_v\|} = \frac{r_u \times r_v}{\sqrt{EG - F^2}}$

Če $\vec{r}(u,v) = (u, v, f(u,v))$

$\Rightarrow z = f(x,y)$ (eksplicitna izražava)

potem $\vec{n} = \frac{(-f_x, -f_y, 1)}{\sqrt{f_x^2 + f_y^2 + 1}}$



N: Parametrizirajte plosked $x^2+y^2-z^2=1$.

CILINDRIČNE KOORDINATE

$$x = r\cdot\cos\varphi$$
$$y = r\cdot\sin\varphi$$
$$z = z$$

$$x^2+y^2-z^2=1$$
$$r^2\cos^2\varphi + r^2\sin^2\varphi - z^2 = 1$$
$$r^2 - z^2 = 1$$
$$r^2 = 1+z^2$$

$r \geq 0: \quad r = \sqrt{1+z^2}$

$\Rightarrow \quad x = \sqrt{1+z^2}\cdot\cos\varphi$, $\quad y = \sqrt{1+z^2}\cdot\sin\varphi$, $\quad z = z$ $\qquad \varphi \in [0, 2\pi)$, $z \in \mathbb{R}$

N: Naj bo S ploskev, podana z enačbo $z = x^2+2y^2$.

a) Zapiši enačbo normalne premice in tangentne ravnine na ploskev v točki $(1,2,9)$.

b) Kaj so koordinatne krivulje pri zgornji parametrizaciji?

c) Pod kakšnim kotom se sekata koordinatni krivulji v točkah $(-\sqrt{2},0,2)$ in $(\frac{1}{2},\frac{1}{4},\frac{3}{8})$?

a) Parametrizacija: $x = u$, $y = v$ $\Rightarrow$ $z = u^2+2v^2$

$\Rightarrow \vec{r}(u,v) = (u, v, u^2+2v^2)$

$\vec{r}_u = (1, 0, 2u)$

$\vec{r}_v = (0, 1, 4v)$

točka $(1,2,9)$: $u_0 = 1$, $v_0 = 2$, $z_0 = 1^2+2\cdot 2^2 = 1+8 = 9$ ✓

$\Rightarrow (1,2,9) = \vec{r}(1,2)$

Normalni vektor: $\overset{\vec{n}}{\parallel}\; r_u \times r_v = (1,0,2u)\times(0,1,4v) = (-2u, -4v, 1)$

$$\begin{matrix} x & y & z \\ 1 & 0 & 2u \\ 0 & 1 & 4v \end{matrix}$$

Normalna premica: $(x,y,z) = (1,2,9) + \eta\cdot\vec{n}(1,2) = (1,2,9)+\eta\cdot(-2,-8,1)$ oz. $\boxed{\frac{x-1}{-2} = \frac{y-2}{-8} = z-9}$

Tangentna ravnina: $\langle (x,y,z)-(1,2,9), (-2,-8,1)\rangle = 0$

$$(x-1)\cdot(-2) + (y-2)\cdot(-8) + (z-9) = 0$$
$$-2x-8y+z = -2-16+9$$
$$-2x-8y+z = -9$$

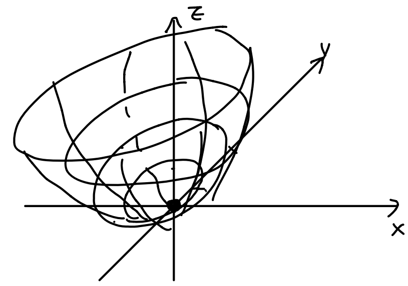
$\boxed{2x+8y-z=9}$

b) $u = C: \quad z = C^2 + 2v^2$ parabola

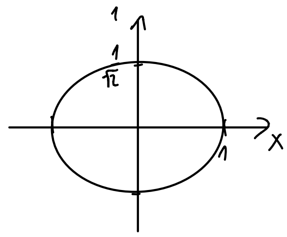
$v = D: \quad z = u^2 + D^2$ parabola

nivojnice: $C = x^2 + 2y^2$

$$1 = \frac{x^2}{\sqrt{C}^2} + \frac{y^2}{\sqrt{\frac{C}{2}}^2}$$



Primer: $C = 1$:



c) $(\sqrt{2}, 0, 2) = \vec{r}(\sqrt{2}, 0)$

$r_u(\sqrt{2}, 0) = (1, 0, 2\sqrt{2})$
$r_v(\sqrt{2}, 0) = (0, 1, 0)$

$r_u \cdot r_v = \|r_u\| \cdot \|r_v\| \cdot \cos\varphi$

$r_u \cdot r_v = (1, 0, 2\sqrt{2}) \cdot (0, 1, 0) = 0$

$\Rightarrow$ sta pravokotna $(\varphi = \frac{\pi}{2})$

$(\frac{1}{2}, \frac{1}{4}, \frac{3}{8}) = \vec{r}(\frac{1}{2}, \frac{1}{4})$

$r_u(\frac{1}{2}, \frac{1}{4}) = (1, 0, 1)$
$r_v(\frac{1}{2}, \frac{1}{4}) = (0, 1, 1)$

$r_u \cdot r_v = \|r_u\| \cdot \|r_v\| \cdot \cos\varphi$

$r_u \cdot r_v = (1, 0, 1) \cdot (0, 1, 1) = 1$

$\|r_u\| = \sqrt{2}, \quad \|r_v\| = \sqrt{2}$

$\Rightarrow \cos\varphi = \frac{1}{2} \quad \Rightarrow \quad \varphi = \frac{\pi}{3}$

DODATNA NALOGA:

(N:) Dana je parametrično zapisana ploskev $\vec{r} = (u\cdot\cos v, u\cdot\sin v, \sqrt{1-u^2})$

a) Zapišite jo v eksplicitni obliki

b) Katera ploskev je to?

c) Pokažite, da so koordinatne krivulje pravokotne povsod, kjer ima to smisel.

d) Pokažite, da je
$$x = \cos^2 t\cos w - \sin t\cos t\sin w$$
$$y = \cos^2 t\sin w + \sin t\cos t\cos w$$
$$z = \sin t$$

zapis ploskve, ki vsebuje prejšnjo ploskev. Zapišite novo ploskev v implicitni obliki. Katera ploskev je to?

e) Izračunajte kot med koordinatnima krivuljama glede na novo parametrizacijo v točki $(1/2, \sqrt{3}/2, 0)$.

f) Dokažite, da koordinatne krivulje glede na novo parametrizacijo niso pravokotne nikjer, kjer ima to smisel.

a) $x = u\cdot\cos v$
$y = u\cdot\sin v$
$z = \sqrt{1-u^2}$

moramo dobiti $z = z(x,y)$

$$z = \sqrt{1-x^2-y^2}$$

$v \in [0, 2\pi)$
$u \in [0, 1]$

b) Zgornja polovica sfere

$z^2 = 1-x^2-y^2$
$x^2+y^2+z^2 = 1, \quad z \geq 0$

c) $\vec{r}_u = (\cos v, \sin v, \frac{1}{2}(1-u^2)^{-\frac{1}{2}}\cdot(-2u)) = (\cos v, \sin v, -\frac{u}{\sqrt{1-u^2}})$ za $u \neq 1$

↑ na spolu in vzporednika

$\vec{r}_v = (-u\sin v, u\cos v, 0)$

$\vec{r}_u \times \vec{r}_v = -u\cdot\cos v\sin v + u\cdot\cos v\cdot\sin v = 0$ (za $u\neq 1$)

d) $x = \cos t\,(\cos t \cdot \cos\omega - \sin t \cdot \sin\omega) = \cos t \cdot \cos(t+\omega)$

$y = \cos t\,(\cos t \sin\omega + \sin t \cdot \cos\omega) = \cos t \cdot \sin(t+\omega)$

$z = \sin t$

$x^2 + y^2 = z^2 = \cos^2 t + \sin^2 t = 1$

cela sfera : $t \in [0, \frac{\pi}{2}]$: $z \in [0,1]$

$t \in [-\frac{\pi}{2}, 0]$: $z \in [-1, 0]$

e) $(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0) \Rightarrow t = 0,\ \omega = \frac{\pi}{3}$

$x_t = -\sin t \cdot \cos(t+\omega) - \cos t \cdot \sin(t+\omega)$

$y_t = -\sin t \cdot \sin(t+\omega) + \cos t \cdot \cos(t+\omega)$

$z_t = \cos t$

$x_\omega = -\cos t \cdot \sin(t+\omega)$

$y_\omega = \cos t \cdot \cos(t+\omega)$

$z_\omega = 0$

φ... kot med vektorjema $(-\frac{\sqrt{3}}{2}, \frac{1}{2}, 1)$ in $(-\frac{\sqrt{3}}{2}, \frac{1}{2}, 0)$

$$\cos\varphi = \frac{(-\frac{\sqrt{3}}{2}, \frac{1}{2}, 1) \cdot (-\frac{\sqrt{3}}{2}, \frac{1}{2}, 0)}{\|(-\frac{\sqrt{3}}{2}, \frac{1}{2}, 1)\| \cdot \|(-\frac{\sqrt{3}}{2}, \frac{1}{2}, 0)\|} = \frac{\frac{3}{4} + \frac{1}{4} + 0}{\sqrt{\frac{3}{4}+\frac{1}{4}+1} \cdot \sqrt{\frac{3}{4}+\frac{1}{4}+0}} = \frac{1}{\sqrt{2} \cdot \sqrt{1}} = \frac{1}{\sqrt{2}}$$

$\Rightarrow \underline{\underline{\varphi = \frac{\pi}{4}}}$

f) $(x_t, y_t, z_t) \cdot (x_\omega, y_\omega, z_\omega) = \cos t \sin t \cancel{\cos(t+\omega)} \sin(t+\omega) + \cos^2 t \sin^2(t+\omega) -$

$- \sin t \cos t \cancel{\cos(t+\omega)} \sin(t+\omega) + \cos^2 t \cos^2(t+\omega) =$

$= \cos^2 t\,(\sin^2(t+\omega) + \cos^2(t+\omega)) =$

$= \cos^2 t \neq 0$ razen, ko $\cos t = 0$ ∈ tam je $(x_\omega, y_\omega, z_\omega) = 0$

↑

Sin J pol

(1): Dana je ploskev $\vec{r} = \begin{bmatrix} u\cdot\cos v \\ u^2\cdot\sin v \\ u^3-u \end{bmatrix}$.

a) Določite, kje so koordinatne krivulje pravokotne.

b) Pri $u=\sqrt{3}$ in $v=\frac{\pi}{3}$ določite točko in tangentno ravnino.

a) $r_u = \begin{bmatrix} \cos v \\ 2u\sin v \\ 3u^2-1 \end{bmatrix}$ $\quad r_v = \begin{bmatrix} -u\sin v \\ u^2\cos v \\ 0 \end{bmatrix}$

$$r_u\cdot r_v = -u\cdot\cos v\cdot\sin v + 2u^3\cos v\sin v =$$
$$= \cos v\cdot\sin v\cdot u\cdot(2u^2-1)$$

$$r_u \perp r_v \Leftrightarrow u=0 \text{ ali } u=\frac{1}{\sqrt{2}} \text{ ali } u=-\frac{1}{\sqrt{2}} \text{ ali } \underbrace{v=k\pi \text{ za } k\in\mathbb{Z} \text{ ali } v=\frac{\pi}{2}+k\pi \text{ za } k\in\mathbb{Z}}_{v=k\frac{\pi}{2} \text{ za } k\in\mathbb{Z}}$$

Točke: $(0,0,0)$, $\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{2\sqrt{2}}\right), \dots$

b) $u=\sqrt{3}$, $v=\frac{\pi}{3}$

$$T = r\left(\sqrt{3},\frac{\pi}{3}\right) = \left(\sqrt{3}\cdot\frac{1}{2}, 3\cdot\frac{\sqrt{3}}{2}, 3\sqrt{3}-\sqrt{3}\right) = \left(\frac{\sqrt{3}}{2}, \frac{3\sqrt{3}}{2}, 2\sqrt{3}\right)$$

$$\vec{r}_u\left(\sqrt{3},\frac{\pi}{3}\right) = \left(\frac{1}{2}, 2\sqrt{3}\cdot\frac{\sqrt{3}}{2}, 3\cdot3-1\right) = \left(\frac{1}{2}, 3, 8\right)$$

$$\vec{r}_v\left(\sqrt{3},\frac{\pi}{3}\right) = \left(-\sqrt{3}\cdot\frac{\sqrt{3}}{2}, 3\cdot\frac{1}{2}, 0\right) = \left(-\frac{3}{2}, \frac{3}{2}, 0\right)$$

$$\vec{n}\left(\sqrt{3},\frac{\pi}{3}\right) = \vec{r}_u\left(\sqrt{3},\frac{\pi}{3}\right) \times \vec{r}_v\left(\sqrt{3},\frac{\pi}{3}\right) = \left(\frac{1}{2},3,8\right)\times\left(-\frac{3}{2},\frac{3}{2},0\right) = \left(-12,-12,\frac{21}{4}\right)$$

$$\frac{4}{3}\vec{n} = (-16,-16,7)$$

želimo vektor s celoštevilskimi koordinatami

$$\left((x,y,z) - \left(\frac{\sqrt{3}}{2}, \frac{3\sqrt{3}}{2}, 2\sqrt{3}\right)\right)\cdot(-16,-16,7) = 0$$

$$\left(x-\frac{\sqrt{3}}{2}\right)\cdot(-16) + \left(y-\frac{3\sqrt{3}}{2}\right)\cdot(-16) + (z-2\sqrt{3})\cdot 7 = 0$$

$$-16x-16y+7z+8\sqrt{3}+8\cdot3\sqrt{3}-14\sqrt{3} = 0$$

$$-16x-16y+7z = -18\sqrt{3}$$

$$\underline{\underline{16x+16y-7z = 18\sqrt{3}}} \leftarrow \text{enačba tangente na ploskev v točki } T$$

(N): Dana je ploskev $x+2y+z^2=1$.

a) Dokaži, da so vse njene tangentne ravnine vzporedne z neko premico.

b) Ali lahko v okolici točke $T(-1,1,0)$ izrazimo z kot funkcijo spremenljivk x in y?

c) Poišči tangentno ravnino na ploskev v točki T.

a) Parametrizacija: $x=1-2y-z^2$ $\qquad r(u,v)=(1-2u-v^2,u,v)$

$y=u$ $\qquad r_u(u,v)=(-2,1,0)$

$z=v$ $\qquad r_v(u,v)=(-2v,0,1)$

$x=1-2u-v^2$

$$\begin{matrix} x & y & z \\ -2 & 1 & 0 \\ -2v & 0 & 1 \end{matrix}$$

$\vec{\nu}(u,v)=(-2,1,0)\times(-2v,0,1)=(1,2,2v)$

Smerni vektor premice: (a,b,c)

Če sta premica in tangentna ravnina vzporedni, je premica pravokotna na $\vec{\nu}$:

$$\vec{\nu}(u,v)\cdot(a,b,c)=0$$
$$(1,2,2v)\cdot(a,b,c)=0$$

$a+2b+2c\cdot v=0$ $\qquad$ Ker v poljuben $\Rightarrow$ $c=0$

$a+2b=0$

$a=-2b$

$\Rightarrow (a,b,c)=(-2b,b,0)=-b\cdot(2,-1,0)$

$\Rightarrow$ Premica s smernim vektorjem $(2,-1,0)$

b) $F(x,y,z)=x+2y+z^2-1$

(a) $F(-1,1,0)=-\cancel{1}+\cancel{2}+0-1=0$ ✓

(b) $JF(x,y,z)=[1,2,2z]$

$JF(-1,1,0)=[\underbrace{1,2}_{x},\underbrace{0}_{y}]$ $\qquad$ $\det F_y=\det[0]=0 \Rightarrow$ NE!

c) $r(u_0,v_0)=(-1,1,0) \Rightarrow u_0=1, v_0=0$ $\qquad$ $-1=1-2\cdot1-0^2$ ✓ leži na ploskvi

$\vec{\nu}(1,0)=(1,2,0)$

$(x-(-1),y-1,z-0)\cdot(1,2,0)=0$ $\qquad$ $\underline{\underline{x+2y=1}}$

$x+1+2y-2=0$

N: Dana je ploskev:

$$\vec{r}(u,v) = \begin{bmatrix} -2\sin u - 10\cos u - 2\sin v - 19\cos v + 14\sin u \sin v \\ 4\sin u + 20\cos u + 7\sin v + 38\cos v - 28\sin u \sin v \\ 5\sin u - 20\cos u + 8\sin v - 38\cos v + 28\sin u \sin v \end{bmatrix}$$

kjer (u,v) preteče neko okolico izhodišča. Pri $u=0$ in $v=0$ določite točko, normalni vektor, normalo in tangentno ravnino.

$$\vec{r}(0,0) = \begin{bmatrix} -29 \\ 58 \\ -58 \end{bmatrix}$$

$$\vec{r}_u(u,v) = \begin{bmatrix} -2\cos u + 10\sin u + 14\cos u \sin v \\ 4\cos u - 20\sin u - 28\cos u \sin v \\ 5\cos u + 20\sin u + 28\cos u \sin v \end{bmatrix} \qquad \vec{r}_u(0,0) = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$$

$$\vec{r}_v(u,v) = \begin{bmatrix} -2\cos v + 19\sin v + 14\sin u \cos v \\ 7\cos v - 38\sin v - 28\sin u \cos v \\ 8\cos v + 38\sin v + 28\sin u \cos v \end{bmatrix} \qquad \vec{r}_v(0,0) = \begin{bmatrix} -2 \\ 7 \\ 8 \end{bmatrix}$$

$$(\vec{r}_u \times \vec{r}_v)(0,0) = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix} \times \begin{bmatrix} -2 \\ 7 \\ 8 \end{bmatrix} = \begin{matrix} x & y & z \\ -2 & 4 & 5 \\ -2 & 7 & 8 \end{matrix} = (4\cdot 8 - 5\cdot 7, -2\cdot 5 + 2\cdot 8, -2\cdot 7 + 2\cdot 4) =$$

$$= (-3, 6, -6) = -3 \cdot \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

$$\vec{n} = \frac{\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}}{\left\| \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \right\|} = \frac{(1,-2,2)}{\sqrt{1^2+(-2)^2+2^2}} = \frac{1}{3}(1,-2,2)$$

NORMALA: $(x,y,z) = (-29, 58, -58) + \lambda \cdot (1,-2,2)$

$$\frac{x+29}{1} = \frac{y-58}{-2} = \frac{z+58}{2}$$

$$2x + \cancel{58} = -y + \cancel{58} = z + \cancel{58}$$

$$\boxed{2x = -y = z}$$

TANGENTNA RAVNINA: $((x,y,z) - (-29, 58, -58)) \cdot (1,-2,2) = 0$

$$(x+29) - 2(y-58) + 2(z+58) = 0$$

$$x - 2y + 2z = -29 - 4\cdot 29 - 4\cdot 29 = -9\cdot 29$$

$$\boxed{x - 2y + 2z = -261}$$

N: Določite normalni vektor in normalo na ploskev $z = \underbrace{x^2 + \frac{18}{y}}_{f(x,y)}$ v točki $T(1,3,z)$.

$x=1, y=3 \Rightarrow z = 1^2 + \frac{18}{3} = 7$

$f_x(x,y) = 2x$

$f_y(x,y) = -18y^{-2}$

$T(1,3,7)$

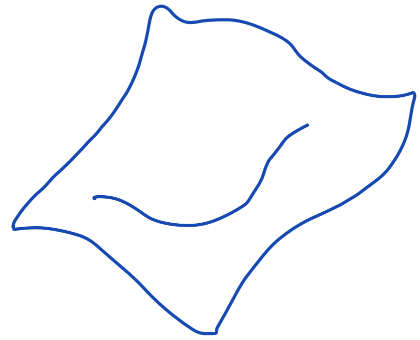
$\vec{n} = \frac{(-f_x, -f_y, 1)}{\sqrt{f_x^2 + f_y^2 + 1}}$ ← ker je ploskev podana eksplicitno

$\vec{n}(1,3) = \frac{(-2,2,1)}{\sqrt{2^2+(-2)^2+1^2}} = \frac{1}{\sqrt{9}}(-2,2,1) = \left(-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$

Normala: $(x,y,z) = (1,3,7) + \lambda \cdot \left(-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$

$\frac{\lambda}{3} = \boxed{\frac{x-1}{-2} = \frac{y-3}{2} = z-7}$

PRVA FUNDAMENTALNA FORMA IN KRIVULJE NA PLOSKVI



PRVA FUNDAMENTALNA FORMA je kvadratna forma, definirana po predpisu

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \mapsto E\alpha^2+2F\alpha\beta+G\beta^2$$

KRIVULJA NA PLOSKVI določena z $u=p(t)$, $v=q(t)$ za $t\in(a,b)$:

dolžina krivulje: $l=\int_a^b\sqrt{E\cdot\dot u^2+2F\dot u\dot v+G\dot v^2}\,dt$

N: Na enotski sferi

$$x=\cos\theta\cos\varphi \qquad y=\cos\theta\sin\varphi \qquad z=\sin\theta$$

izračunajte dolžino poti, določene s:

$$\varphi=\frac{1}{2}\ln\frac{1+t}{1-t} \qquad \theta=\arccos t \qquad 0\le t\le\frac{1}{2}$$

$$\vec r(\varphi,\theta)=(\cos\varphi\cos\theta,\ \sin\varphi\sin\theta,\ \sin\theta)$$

$$\vec r_\varphi(\varphi,\theta)=(-\sin\varphi\cos\theta,\ \cos\varphi\cos\theta,\ 0)$$

$$\vec r_\theta(\varphi,\theta)=(-\cos\varphi\sin\theta,\ -\sin\varphi\sin\theta,\ \cos\theta)$$

$$E=\vec r_\varphi\cdot\vec r_\varphi=\cos^2\theta=(\cos\theta)^2=(\cos(\arccos t))^2=t^2$$

$$F=\vec r_\varphi\cdot\vec r_\theta=\sin\varphi\cdot\cos\theta\cdot\cos\varphi\cdot\sin\theta-\cos\varphi\cdot\cos\theta\cdot\sin\varphi\cdot\sin\theta=0$$

$$G=\vec r_\theta\cdot\vec r_\theta=\sin^2\theta+\cos^2\theta=1$$

$$\dot\varphi(t)=\frac{1}{2}\cdot\frac{1}{\frac{1+t}{1-t}}\cdot\left(\dot{\frac{1+t}{1-t}}\right)=\frac{1}{2}\,\frac{\cancel{1-t}}{1+t}\cdot\frac{1\cdot(1-t)-(-1)\cdot(1+t)}{(1-t)^{\cancel{2}}}=\frac{1}{2}\,\frac{1-\cancel{t}+1+\cancel{t}}{(1+t)(1-t)}=\frac{1}{\cancel{2}}\,\frac{\cancel{2}}{1-t^2}=\frac{1}{1-t^2}$$

$$\dot\theta(t)=-\frac{1}{\sqrt{1-t^2}}$$

$$l=\int_0^{\frac{1}{2}}\sqrt{t^2\cdot\frac{1}{(1-t^2)^2}+0+1\cdot\frac{1}{1-t^2}}\,dt=\int_0^{\frac{1}{2}}\frac{1}{1-t^2}\cdot\sqrt{\cancel{t^2}+(1-\cancel{t^2})}\,dt=\int_0^{\frac{1}{2}}\frac{1}{1-t^2}\,dt\overset{*}{=}$$

$$=\frac{1}{2}\int_0^{\frac{1}{2}}\frac{1}{1-t}\,dt+\frac{1}{2}\int_0^{\frac{1}{2}}\frac{1}{1+t}\,dt=\frac{1}{2}\left(-\ln(1-t)+\ln(1+t)\right)\Big|_0^{\frac{1}{2}}=\underline{\underline{\frac{1}{2}\ln 3}}$$

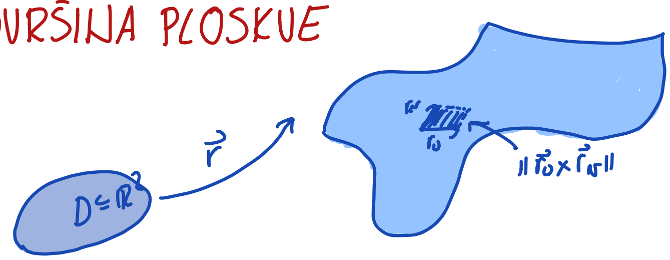
$$*\ \frac{1}{1-t^2}=\frac{A}{1-t}+\frac{B}{1+t}$$

$$1=A(1+t)+B(1-t)$$

$$1=A+B+t\cdot(A-B)$$

$$\left.\begin{array}{l}A+B=1\\ A-B=0\end{array}\right\}A=B=\frac{1}{2}$$

POVRŠINA PLOSKVE



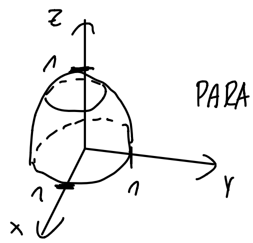
1: Izračunajte površino tistega dela ploskve $x^2+y^2+z=1$, ki leži nad ravnino xy.

Nad ravnino $xy \Rightarrow z \geq 0$

$z = 1-(x^2+y^2)$

$$\left.\begin{array}{l} z \geq 0 \\ z = 1-(x^2+y^2) \end{array}\right\} \quad 0 \leq x^2+y^2 \leq 1$$

Skica:



CILINDRIČNE KOORDINATE (če želimo "lepo" opisati območje D: zaradi integriranja po D)

$$x = r\cdot\cos\varphi \qquad z = 1-(x^2+y^2) = 1-r^2$$
$$y = r\cdot\sin\varphi$$
$$z = z \qquad 0 \leq r^2 \leq 1 \Rightarrow 0 \leq r \leq 1$$

$$\vec{r}(r,\varphi) = (r\cdot\cos\varphi,\ r\cdot\sin\varphi,\ 1-r^2)$$

$$D = [0,1] \times [0,2\pi)$$

$r \in [0,1]$ $\qquad$ $\varphi \in [0,2\pi)$

$$\vec{r}_r(r,\varphi) = (\cos\varphi,\ \sin\varphi,\ -2r)$$

$$\vec{r}_\varphi(r,\varphi) = (-r\cdot\sin\varphi,\ r\cdot\cos\varphi,\ 0)$$

$$E = \cos^2\varphi + \sin^2\varphi + 4r^2 = 1+4r^2$$
$$F = -r\cdot\cos\varphi\cdot\sin\varphi + r\cdot\cos\varphi\cdot\sin\varphi + 0 = 0 \qquad E\cdot G - F^2 = r^2\cdot(1+4r^2)$$
$$G = r^2\cdot\sin^2\varphi + r^2\cdot\cos^2\varphi = r^2$$

$$P = \int_0^{2\pi} d\varphi \int_0^1 \sqrt{r^2\cdot(1+4r^2)}\, dr = 2\pi \int_0^1 r\sqrt{1+4r^2}\, dr \underset{\uparrow}{=} 2\pi \int_1^5 \cancel{r}\cdot\sqrt{t}\cdot\frac{dt}{8\cancel{r}} = \frac{2\pi}{8}\,\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_1^5 = \frac{\pi}{6}\cdot(5\sqrt{5}-1)$$

$t = 1+4r^2$

$dt = 8r\,dr$

(D:) Izračunajte površino naslednjega dela helikoida:

$$x = \rho \cdot \cos\varphi$$
$$y = \rho \cdot \sin\varphi \qquad 0 < \varphi < \rho < 2\pi$$
$$z = \varphi$$

$$\vec{r}(\rho,\varphi) = (\rho\cdot\cos\varphi, \rho\cdot\sin\varphi, \varphi)$$
$$\vec{r}_\rho(\rho,\varphi) = (\cos\varphi, \sin\varphi, 0)$$
$$\vec{r}_\varphi(\rho,\varphi) = (-\rho\sin\varphi, \rho\cos\varphi, 1)$$

$$E = \cos^2\varphi + \sin^2\varphi = 1$$
$$F = -\rho\cos\varphi\sin\varphi + \rho\cos\varphi\sin\varphi = 0$$
$$G = \rho^2\sin^2\varphi + \rho^2\cos^2\varphi + 1 = \rho^2 + 1$$

$$EG - F^2 = \rho^2 + 1$$

$$P = \int_0^{2\pi} d\rho \int_0^{\rho} \sqrt{\rho^2+1}\, d\varphi = \int_0^{2\pi} \sqrt{\rho^2+1}\cdot \varphi \Big|_{\varphi=0}^{\rho} d\rho = \int_0^{2\pi} \rho\cdot\sqrt{\rho^2+1}\, d\rho \underset{\uparrow}{=} \int_1^{4\pi^2+1} \cancel{\rho}\cdot\sqrt{t}\cdot\frac{dt}{2\cancel{\rho}} =$$

$$t = \rho^2 + 1$$
$$dt = 2\rho\, d\rho$$

$$= \frac{1}{\cancel{2}}\cdot \frac{t^{\frac{3}{2}}}{\frac{3}{\cancel{2}}}\Big|_{t=1}^{4\pi^2+1} = \frac{1}{3}\left((4\pi^2+1)^{\frac{3}{2}} - 1\right)$$

DRUGA FUNDAMENTALNA FORMA IN UKRIVLJENOST PLOSKVE

7.4.2021

$\vec{r}: D \to \mathbb{R}^3$ parametrizacija ploskve, $D \subseteq \mathbb{R}^2$

KOEFICIENTI DRUGE FUNDAMENTALNE FORME:

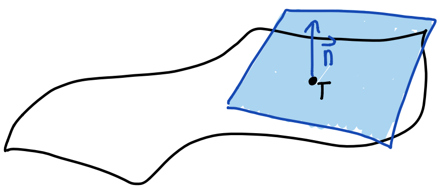
$$L(u,v) = \vec{r}_{uu} \cdot \vec{n} = \frac{[\vec{r}_u, \vec{r}_v, \vec{r}_{uu}]}{\sqrt{EG-F^2}}$$

$$M(u,v) = \vec{r}_{uv} \cdot \vec{n} = \frac{[\vec{r}_u, \vec{r}_v, \vec{r}_{uv}]}{\sqrt{EG-F^2}}$$

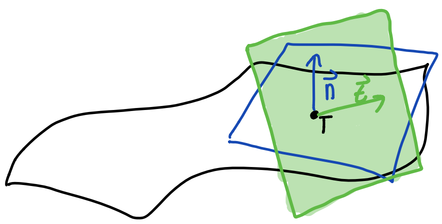
$$N(u,v) = \vec{r}_{vv} \cdot \vec{n} = \frac{[\vec{r}_u, \vec{r}_v, \vec{r}_{vv}]}{\sqrt{EG-F^2}}$$

Druga fundamentalna forma: $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \mapsto L\alpha^2 + 2M\alpha\beta + N\beta^2$

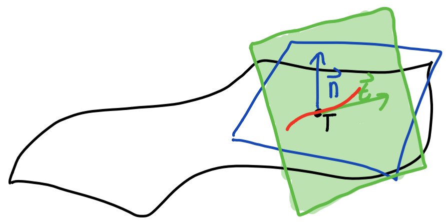
UKRIVLJENOSTI PLOSKVE



ploskev in tangentna ravnina na ploskev v neki točki T



za vsak vektor $\vec{t}$ na tangentni ravnini opazujemo ravnino, ki jo napenjata vektorja $\vec{n}$ in $\vec{t}$



presek te ravnine s ploskvijo nam poda krivuljo na ploskvi skozi točko T
NORMALNI PRESEK

opazujemo fleksijske ukrivljenosti takih krivulj (pri različnih vektorjih $\vec{t} = \underbrace{\alpha \cdot \vec{r}_u + \beta \cdot \vec{r}_v}$)

vsak vektor $\vec{t}$ na tangentni ravnini je take oblike

$$K = \frac{L\alpha^2 + 2M\alpha\beta + N\beta^2}{E\alpha^2 + 2F\alpha\beta + G\beta^2}$$ ← UKRIVLJENOST NORMALNEGA PRESEKA

Najmanjši in največji izmed njih pravimo GLAVNI UKRIVLJENOSTI $\mathcal{K}_1$ in $\mathcal{K}_2$.

Smeri tangentnih vektorjev krivulj, pri katerih se pojavita glavni ukrivljenosti, sta GLAVNI SMERI.

Izračun glavnih ukrivljenosti:

$$\det\left(\begin{bmatrix} L & M \\ M & N \end{bmatrix} - \mathcal{K}\begin{bmatrix} E & F \\ F & G \end{bmatrix}\right) = 0$$

← lastni vrednosti matrike $\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1}\begin{bmatrix} L & M \\ M & N \end{bmatrix}$

Izračun pripadajočih glavnih smeri: $\frac{\alpha}{\beta} = -\frac{M - \mathcal{K}F}{L - \mathcal{K}E} \to \vec{t} = \alpha\vec{r}_u + \beta\vec{r}_v$

↖ lastna vektorja zgornje matrike

KLASIFIKACIJA TOČK NA PLOSKVI GLEDE NA GLAVNE UKRIVLJENOSTI

- eliptična: $\kappa_1 \cdot \kappa_2 > 0$
- krogelna: $\kappa_1, \kappa_2 > 0$ in $\kappa_1 = \kappa_2$
- hiperbolična: $\kappa_1 \cdot \kappa_2 < 0$
- parabolična: $\kappa_1 \cdot \kappa_2 = 0$ (primer: plašč valja)
- planarna: $\kappa_1 = \kappa_2 = 0$ (lokalno ravnina)

GAUSSOVA UKRIVLJENOST: $K = \kappa_1 \cdot \kappa_2 = \det\left(\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} L & M \\ M & N \end{bmatrix}\right) = \dfrac{LN - M^2}{EG - F^2}$

POVPREČNA UKRIVLJENOST: $H = \dfrac{\kappa_1 + \kappa_2}{2} = \dfrac{1}{2} \operatorname{tr}\left(\dfrac{1}{EG - F^2} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \begin{bmatrix} L & M \\ M & N \end{bmatrix}\right) = \dfrac{GL - 2FM + EN}{2(EG - F^2)}$

(10:) Dana je ploskev $\vec{r}(u,v) = (u\cos v, u\sin v, v)$.

a) Izračunaj glavni ukrivljenosti in glavni smeri v točki $\vec{r}(1,\frac{\pi}{2})$.

b) Izračunaj Gaussovo in povprečno ukrivljenost v točki $\vec{r}(1,\frac{\pi}{2})$.

v točki $r(1,\frac{\pi}{2})$:

$$r_u(u,v) = (\cos v, \sin v, 0) \qquad r_u(1,\tfrac{\pi}{2}) = (0,1,0)$$
$$r_v(u,v) = (-u\cdot\sin v, u\cdot\cos v, 1) \qquad r_v(1,\tfrac{\pi}{2}) = (-1,0,1)$$
$$r_{uu}(u,v) = (0,0,0) \qquad r_{uu}(1,\tfrac{\pi}{2}) = (0,0,0)$$
$$r_{uv}(u,v) = (-\sin v, \cos v, 0) \qquad r_{uv}(1,\tfrac{\pi}{2}) = (-1,0,0)$$
$$r_{vv}(u,v) = (-u\cos v, -u\sin v, 0) \qquad r_{vv}(1,\tfrac{\pi}{2}) = (0,-1,0)$$

$E = r_u\cdot r_u = 1$
$F = r_u\cdot r_v = 0$
$G = r_v\cdot r_v = 2$

$\sqrt{EG-F^2} = \sqrt{2}$

$r_u\times r_v = (0,1,0)\times(-1,0,1) = (1,0,1)$

$\|r_u\times r_v\| = \sqrt{2}$

$\vec{n} = \frac{(1,0,1)}{\sqrt{2}}$

$L = r_{uu}\cdot n = (0,0,0)\cdot\frac{(1,0,1)}{\sqrt{2}} = 0$

$M = r_{uv}\cdot n = (-1,0,0)\cdot\frac{(1,0,1)}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$

$N = r_{vv}\cdot n = (0,-1,0)\cdot\frac{(1,0,1)}{\sqrt{2}} = 0$

a)
$$\begin{bmatrix} L & M \\ M & N \end{bmatrix} - \lambda\begin{bmatrix} E & F \\ F & G\end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0\end{bmatrix} - \lambda\cdot\begin{bmatrix} 1 & 0 \\ 0 & 2\end{bmatrix} = \begin{bmatrix} -\lambda & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -2\lambda\end{bmatrix}$$

$$\det\begin{bmatrix} -\lambda & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\lambda\end{bmatrix} = 2\lambda^2 - \frac{1}{2} = 0 \Rightarrow \lambda^2 = \frac{1}{4} \Rightarrow \lambda_{1,2} = \pm\frac{1}{2}$$

Glavni ukrivljenosti: $\lambda_1 = \frac{1}{2}$ $\lambda_2 = -\frac{1}{2}$

Glavni smeri:

$$\lambda = \frac{1}{2}: \quad \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0\end{bmatrix}\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 2\end{bmatrix}\begin{bmatrix}\alpha \\ \beta\end{bmatrix} \Rightarrow \begin{bmatrix} -\frac{1}{\sqrt{2}}\beta \\ -\frac{1}{\sqrt{2}}\alpha\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\alpha \\ \beta\end{bmatrix} \Rightarrow \underbrace{\begin{matrix}\beta = -\frac{1}{\sqrt{2}}\alpha \\ \frac{1}{2}\alpha = -\frac{1}{\sqrt{2}}\beta\end{matrix}}_{\text{ekvivalentni enačbi}}$$

$$\vec{t} = -\sqrt{2}\cdot\beta\cdot r_u + \beta\cdot r_v = \beta\cdot(-\sqrt{2}\,r_u + r_v)$$

Dovolj samo $\vec{t} = -\sqrt{2}\,r_u + r_v = -\sqrt{2}(0,1,0) + (-1,0,1) = (-1,-\sqrt{2},1)$

$$\lambda = -\frac{1}{2}: \quad \frac{\alpha}{\beta} = -\frac{M-\lambda F}{L-\lambda E} = -\frac{-\frac{1}{\sqrt{2}} - (-\frac{1}{2})\cdot 0}{0-(-\frac{1}{2})\cdot 1} = \frac{\frac{1}{\sqrt{2}}}{\frac{1}{2}} = \sqrt{2} \Rightarrow \alpha = \sqrt{2}\beta$$

$$\vec{t} = \sqrt{2}\cdot r_u + r_v = \sqrt{2}(0,1,0) + (-1,0,1) = (-1,\sqrt{2},1)$$

b) Gaussova ukrivljenost: $K = \lambda_1\cdot\lambda_2 = \frac{1}{2}\cdot(-\frac{1}{2}) = -\frac{1}{4}$ drugi način: $K = \frac{LN-M^2}{EG-F^2} = \ldots = -\frac{1}{4}$

Povprečna ukrivljenost: $H = \frac{1}{2}(\lambda_1+\lambda_2) = 0$ drugi način: $H = \frac{GL-2FM+EN}{2(EG-F^2)}$

N: Dana je ploskev $\vec{r}(u,v) = (u^2+v,\ u+v^2,\ uv)$.

a) Dokažite, da na njej obstaja natanko ena točka, ki ima koordinati $y=-2$ in $z=0$. Izračunajte še koordinato x te točke.

b) Izračunajte Gaussovo in povprečno ukrivljenost ploskve v tej točki. Klasificirajte točko.

a)
$$\begin{aligned} x &= u^2+v \\ y &= u+v^2 = -2 \\ z &= u\cdot v = 0 \end{aligned}$$

če $u=0$: $y = 0+v^2 = -2$ $\Rightarrow\Leftarrow$

če $v=0$: $y = u+0^2 = -2$ $\Rightarrow u=-2$

$\Rightarrow$ $u=-2$, $v=0$ in $x = u^2+v = (-2)^2+0 = 4$

b)
$$r_u = (2u, 1, v) \qquad r_u(-2,0) = (-4,1,0)$$
$$r_v = (1, 2v, u) \qquad r_v(-2,0) = (1,0,-2)$$

$$(r_u \times r_v)(-2,0) = (-4,1,0)\times(1,0,-2) = (-2,-8,-1)$$
$$\|r_u \times r_v\| = \sqrt{4+64+1} = \sqrt{69}$$
$$n = \frac{(-2,-8,-1)}{\sqrt{69}}$$

$$r_{uu} = (2,0,0) \qquad r_{uu}(-2,0) = (2,0,0)$$
$$r_{uv} = (0,0,1) \qquad r_{uv}(-2,0) = (0,0,1)$$
$$r_{vv} = (0,2,0) \qquad r_{vv}(-2,0) = (0,2,0)$$

$$E = r_u\cdot r_u = (-4,1,0)\cdot(-4,1,0) = 17$$
$$F = r_u\cdot r_v = (-4,1,0)\cdot(1,0,-2) = -4$$
$$G = r_v\cdot r_v = (1,0,-2)\cdot(1,0,-2) = 5$$

$$L = r_{uu}\cdot n = \frac{1}{\sqrt{69}}\cdot(2,0,0)\cdot(-2,-8,-1) = -\frac{4}{\sqrt{69}}$$
$$M = r_{uv}\cdot n = \frac{1}{\sqrt{69}}\cdot(0,0,1)\cdot(-2,-8,-1) = -\frac{1}{\sqrt{69}}$$
$$N = r_{vv}\cdot n = \frac{1}{\sqrt{69}}\cdot(0,2,0)\cdot(-2,-8,-1) = -\frac{16}{\sqrt{69}}$$

$$K = \frac{LN-M^2}{EG-F^2} = \frac{\frac{4\cdot16}{69}-\frac{1}{69}}{17\cdot5-16} = \frac{1}{69}\cdot\frac{63}{69} = \frac{\cancel{3}\cdot\cancel{3}\cdot7}{\cancel{3}\cdot23\cdot\cancel{3}\cdot23} = \frac{7}{529}$$

$$H = \frac{GL-2FM+EN}{2(EG-F^2)} = \frac{-\frac{20}{\sqrt{69}}-\frac{8}{\sqrt{69}}-\frac{17\cdot16}{\sqrt{69}}}{2\cdot(17\cdot5-16)} = -\frac{4}{2\sqrt{69}}\cdot\frac{5+2+68}{69} = -\frac{2}{69\sqrt{69}}\cdot75 = -\frac{50}{23\sqrt{69}}$$

$K = \kappa_1\cdot\kappa_2 = \frac{7}{529}$ $\Rightarrow$ $\kappa_1\cdot\kappa_2 > 0$ $\Rightarrow$ Točka $(4,-2,0)$ je eliptična.

Ni pa krogelna, saj bi sicer veljalo $H = \frac{1}{2}(\kappa_1+\kappa_2) = \frac{1}{2}\cdot2\kappa_1 = \kappa_1 = \kappa_2$ $\Rightarrow$ $K = H^2$

$$\frac{7}{\cancel{529}} = \frac{50^2}{\cancel{23^2}\cdot69}$$

$$7\cdot69 = 50^2$$
$\Rightarrow\Leftarrow$

(1): Dana je ploskev $4z = 7x^2 - 6xy\sqrt{3} + 13y^2$.

a) Izračunaj glavni ukrivljenosti in smeri v točki $(0,0,0)$.

b) -||- $(1,2,4)$.

c) Izračunaj ukrivljenost normalnega preseka v točki $(0,0,0)$ v smeri tangentnega vektorja $(2,3,0)$.

d) Izračunaj ukrivljenost preseka z ravnino $x=z$ v točki $(0,0,0)$.

$r(u,v) = (u, v, \frac{1}{4}(7u^2 - 6uv\sqrt{3} + 13v^2))$ $\qquad r_{uu}(u,v) = (0,0,\frac{7}{2})$

$r_u(u,v) = (1,0,\frac{1}{4}(14u - 6v\sqrt{3}))$ $\qquad r_{uv}(u,v) = (0,0,-\frac{3\sqrt{3}}{2})$

$r_v(u,v) = (0,1,\frac{1}{4}(-6u\sqrt{3} + 26v))$ $\qquad r_{vv}(u,v) = (0,0,\frac{13}{2})$

a) $(0,0,0) = r(0,0)$

$r_u(0,0) = (1,0,0)$
$r_v(0,0) = (0,1,0)$
$r_{uu}(0,0) = (0,0,\frac{7}{2})$
$r_{uv}(0,0) = (0,0,-\frac{3\sqrt{3}}{2})$
$r_{vv}(0,0) = (0,1,\frac{13}{2})$

$r_u \times r_v = (1,0,0)\times(0,1,0) = (0,0,1)$
$\|r_u \times r_v\| = 1$
$n = (0,0,1)$

$E = r_u \cdot r_u = 1$ $\qquad L = r_{uu}\cdot n = \frac{7}{2}$
$F = r_u \cdot r_v = 0$ $\qquad M = r_{uv}\cdot n = -\frac{3\sqrt{3}}{2}$
$G = r_v \cdot r_v = 1$ $\qquad N = r_{vv}\cdot n = \frac{13}{2}$

$\sqrt{EG-F^2} = 1$

$$\det\left(\begin{bmatrix} \frac{7}{2} & -\frac{3\sqrt{3}}{2} \\ -\frac{3\sqrt{3}}{2} & \frac{13}{2}\end{bmatrix} - \kappa\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = \begin{vmatrix} \frac{7}{2}-\kappa & -\frac{3\sqrt{3}}{2} \\ -\frac{3\sqrt{3}}{2} & \frac{13}{2}-\kappa\end{vmatrix} = (\tfrac{7}{2}-\kappa)(\tfrac{13}{2}-\kappa) - \frac{9\cdot 3}{4} = \frac{7\cdot 13}{4} - (\tfrac{7}{2}+\tfrac{13}{2})\kappa + \kappa^2 - \frac{27}{4} =$$

$$= \frac{91}{4} - \frac{27}{4} - \frac{20}{2}\kappa + \kappa^2 = \kappa^2 - 10\kappa + \frac{64}{4} = \kappa^2 - 10\kappa + 16 = (\kappa-2)(\kappa-8)$$

Glavni ukrivljenosti: $\kappa_1 = 2$ in $\kappa_2 = 8$

Glavni smeri: $\kappa_1 = 2$: $\frac{\alpha}{\beta} = -\frac{-\frac{3\sqrt{3}}{2}}{\frac{7}{2}-2} = -\frac{-\frac{3\sqrt{3}}{2}}{\frac{3}{2}} = \sqrt{3}$ $\Rightarrow$ $\alpha\cdot r_u + \beta\cdot r_v = \sqrt{3}\beta r_u + \beta r_v$

$\Rightarrow t = \sqrt{3}\cdot(1,0,0) + (0,1,0) = (\sqrt{3},1,0)$

$\kappa_2 = 8$: $\frac{\alpha}{\beta} = -\frac{-\frac{3\sqrt{3}}{2}}{\frac{7}{2}-8} = -\frac{-\frac{3\sqrt{3}}{2}}{-\frac{9}{2}} = -\frac{\sqrt{3}}{3}$ $\Rightarrow$ $\alpha r_u + \beta r_v = -\frac{\sqrt{3}}{3}\beta r_u + \beta r_v = \frac{\beta}{3}(-\sqrt{3}r_u + 3r_v)$

$\Rightarrow t = -\sqrt{3}(1,0,0) + 3\cdot(0,1,0) = (-\sqrt{3},3,0)$

b) DN

c) $(2,3,0) = 2\cdot(1,0,0) + 3\cdot(0,1,0) = 2\cdot r_u(0,0) + 3\cdot r_v(0,0) \Rightarrow \alpha = 2,\ \beta = 3$

Iz a) primera vemo: $E=1,\ F=0,\ G=1,\ L=\frac{7}{2},\ M=-\frac{3\sqrt{3}}{2},\ N=\frac{13}{2}$

$$K = \frac{L\alpha^2+2M\alpha\beta+N\beta^2}{E\alpha^2+2F\alpha\beta+G\beta^2} = \frac{\frac{7}{2}\cdot 2^2+2\cdot\left(-\frac{3\sqrt{3}}{2}\right)\cdot 2\cdot 3+\frac{13}{2}\cdot 3^2}{1\cdot 2^2+2\cdot 0\cdot 2\cdot 3+1\cdot 3^2} = \frac{1}{2}\cdot\frac{28-36\sqrt{3}+117}{4+9} = \frac{145-36\sqrt{3}}{26}$$

d) Iščemo $\underbrace{\text{vektor na ravnini } x=z}_{(a,b,a)}$, ki je hkrati na tangentni ravnini v točki $r(u,v)$.

↳ pravokoten je na normalo n

$(a,b,a)\cdot n = 0$

V točki $\vec{r}(0,0)$: $(a,b,a)\cdot(0,0,1) = 0 \Rightarrow a = 0$

$\Rightarrow (0,b,0) = b\cdot(0,1,0)$

$t = (0,1,0) = 0\cdot r_u + 1\cdot r_v$

Okrogljenost:
$$K = \frac{L\cdot 0^2+2M\cdot 0\cdot 1+N\cdot 1^2}{E\cdot 0^2+2F\cdot 0\cdot 1+G\cdot 1^2} = \frac{N}{G} = \underline{\underline{\frac{13}{2}}}$$

19: Dana je ploskev:

$$\vec r(u,v) = \begin{bmatrix} -2\sin u - 10\cos u - 2\sin v - 19\cos v + 14\sin u \sin v \\ 4\sin u + 20\cos u + 7\sin v + 38\cos v - 28\sin u\sin v \\ 5\sin u - 20\cos u + 8\sin v - 38\cos v + 28\sin u \sin v \end{bmatrix}$$

kjer (u,v) preteče neko okolico izhodišča. Pri $u=0$ in $v=0$ izračunajte ukrivljenost normalnega preseka v smeri tangentnega vektorja $\vec w = (-4,5,7)$

Vemo (naloga zgoraj): $\vec r(0,0) = \begin{bmatrix} -29 \\ 58 \\ -58 \end{bmatrix}$ $\quad \vec r_u(0,0) = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$ $\quad \vec r_v(0,0) = \begin{bmatrix} -2 \\ 7 \\ 8 \end{bmatrix}$

$$\vec n = \frac{1}{3}(1,-2,2)$$

$$\vec r_u(u,v) = \begin{bmatrix} -2\cos u + 10\sin u + 14\cos u \sin v \\ 4\cos u - 20\sin u - 28\cos u \sin v \\ 5\cos u + 20\sin u + 28\cos u \sin v \end{bmatrix}$$

$$\vec r_v(u,v) = \begin{bmatrix} -2\cos v + 19\sin v + 14\sin u\cos v \\ 7\cos v - 38\sin v - 28\sin u \cos v \\ 8\cos v + 38\sin v + 28\sin u\cos v \end{bmatrix}$$

Izračunamo: $\vec r_{uu}(u,v) = \begin{bmatrix} 2\sin u + 10\cos u - 14\sin u\sin v \\ -4\sin u - 20\cos u + 28\sin u \sin v \\ -5\sin u + 20\cos u - 28\sin u\sin v \end{bmatrix}$ $\qquad \vec r_{uu}(0,0) = \begin{bmatrix} 10 \\ -20 \\ 20 \end{bmatrix}$

$\vec r_{uv}(u,v) = \begin{bmatrix} 14\cos u\cos v \\ -28\cos u\cos v \\ 28\cos u \cos v \end{bmatrix}$ $\qquad \vec r_{uv}(0,0) = \begin{bmatrix} 14 \\ -28 \\ 28 \end{bmatrix}$

$\vec r_{vv}(u,v) = \begin{bmatrix} 2\sin v + 19\cos v - 14\sin u\sin v \\ -7\sin v - 38\cos v + 28\sin u\sin v \\ -8\sin v + 38\cos v - 28\sin u\sin v \end{bmatrix}$ $\qquad \vec r_{vv}(0,0) = \begin{bmatrix} 19 \\ -38 \\ 38 \end{bmatrix}$

$E = 45 \quad F = 72 \quad G = 117 \quad L = -30 \quad M = -42 \quad N = -57$

$$w = (-4,5,7) = \alpha \cdot \vec r_u(0,0) + \beta \cdot \vec r_v(0,0) = \alpha\cdot(-2,4,5) + \beta\cdot(-2,7,8)$$

$-4 = -2\alpha - 2\beta \;\rightarrow\; \alpha + \beta = 2 \;\rightarrow\; \beta = 2 - \alpha \qquad \beta = 2 - 3 = -1$

$5 = 4\alpha + 7\beta \;\rightarrow\; 5 = 4\alpha + 7\cdot(2-\alpha)$

$5 = 4\alpha + 14 - 7\alpha$

$3\alpha = 9$

$\alpha = 3 \qquad \Rightarrow \boxed{\alpha = 3,\ \beta = -1}$

$7 = 5\alpha + 8\beta = 5\cdot 3 + 8\cdot(-1) = 15 - 8 = 7$ ✓ (w res leži na tangentni ravnini)

$$K = \frac{-30\cdot 9 + 2\cdot 3\cdot(-42) - 57}{45\cdot 9 - 2\cdot 3\cdot 72 + 117} = \frac{3(-90 + 84 - 19)}{9(45 - 48 + 13)} = \frac{-25}{3\cdot 10} = -\frac{5}{6}$$

Pri $u=0$ in $v=0$ izračunajte glavni ukrivljenosti in določite glavni smeri. Točko $\vec{r}(0,0)$ klasificirajte.

$$\det\left(\begin{bmatrix}-30 & -42\\ -42 & -57\end{bmatrix} - \lambda\begin{bmatrix}45 & 72\\ 72 & 117\end{bmatrix}\right) = \begin{vmatrix}-30-\lambda\cdot 45 & -42-\lambda\cdot 72\\ -42-\lambda\cdot 72 & -57-\lambda\cdot 117\end{vmatrix} =$$

$$= \begin{vmatrix}-15\cdot(2+3\lambda) & -6\cdot(7+12\cdot\lambda)\\ -6\cdot(7+12\cdot\lambda) & -3\cdot(19+13\cdot 3\cdot\lambda)\end{vmatrix} = 9\cdot\begin{vmatrix}5\cdot(2+3\lambda) & 2\cdot(7+12\lambda)\\ 2\cdot(7+12\lambda) & 19+13\cdot 3\lambda\end{vmatrix} =$$

$$= 9\cdot\left(5\cdot(2+3\lambda)(19+13\cdot 3\lambda) - 4\cdot(7+12\lambda)^2\right) = 9\cdot\left(5\cdot(38+135\lambda+117\lambda^2) - 4\cdot(49+168\lambda+144\lambda^2)\right) =$$

$$= 9\cdot(-6+3\lambda+9\lambda^2) = 27\cdot(3\lambda^2+\lambda-2) = 0 \quad \Leftrightarrow \quad \lambda_{1,2} = \frac{-1\pm\sqrt{1+4\cdot 2\cdot 3}}{6} = \frac{-1\pm\sqrt{25}}{6} = \frac{-1\pm 5}{6}$$

Glavni ukrivljenosti: $\lambda_1 = -1 \quad \lambda_2 = \frac{2}{3}$

Glavni smeri: $\bullet\ \lambda_1 = -1$:

$$\frac{\alpha}{\beta} = -\frac{M-\lambda F}{L-\lambda E} = -\frac{-42-(-1)\cdot 72}{-30-(-1)\cdot 45} = -\frac{72-42}{45-30} = -\frac{30}{15} = -2$$

$$\alpha\cdot r_u + \beta\cdot r_v = -2\beta\cdot(-2,4,5) + \beta\cdot(-2,7,8)$$

$$\Rightarrow -2(-2,4,5) + (-2,7,8) = (2,-1,-2)$$

$\bullet\ \lambda_2 = \frac{2}{3}$:

$$\frac{\alpha}{\beta} = -\frac{M-\lambda F}{L-\lambda E} = -\frac{-42-\frac{2}{3}\cdot 72}{-30-\frac{2}{3}\cdot 45} = -\frac{-42-48}{-30-30} = -\frac{-90}{-60} = -\frac{3}{2}$$

$$\alpha\cdot r_u + \beta\cdot r_v = -\frac{3}{2}\beta\cdot(-2,4,5) + \beta\cdot(-2,7,8)$$

$$\Rightarrow -3\cdot(-2,4,5) + 2\cdot(-2,7,8) = (6,-12,-15) + (-4,14,16) = (2,2,1)$$

Ker je $\lambda_1\cdot\lambda_2 < 0$, je točka $\vec{r}(0,0)$ hiperbolična.

Pri $u=0$ in $v=0$ izračunajte Gaussovo in povprečno ukrivljenost.

$$K = \lambda_1\cdot\lambda_2 = -\frac{2}{3}$$

$$H = \frac{1}{2}(\lambda_1+\lambda_2) = \frac{1}{2}\left(-1+\frac{2}{3}\right) = \frac{1}{2}\cdot\left(-\frac{1}{3}\right) = -\frac{1}{6}$$

(N:) Na ploskvi $z=x^3y^2$ klasificirajte točko, kjer je $x=y=1$, ter tam določite Gaussovo in povprečno ukrivljenost.

$\vec{r}(x,y) = (x,y,x^3y^2)$

$\vec{r}(u,v) = (u,v,u^3v^2)$

$\vec{r}_u = (1,0,3u^2v^2)$ $\quad$ $\vec{r}_u(1,1) = (1,0,3)$ $\quad$ $\vec{r}_u(1,1) \times \vec{r}_v(1,1) = (-3,-2,1)$

$\vec{r}_v = (0,1,2u^3v)$ $\quad$ $\vec{r}_v(1,1) = (0,1,2)$ $\quad$ $\vec{n}(1,1) = \frac{1}{\sqrt{9+4+1}}(-3,-2,1) = \frac{1}{\sqrt{14}}(-3,-2,1)$

$\vec{r}_{uu} = (0,0,6uv^2)$ $\quad$ $\vec{r}_{uu}(1,1) = (0,0,6)$

$\vec{r}_{uv} = (0,0,6u^2v)$ $\quad$ $\vec{r}_{uv}(1,1) = (0,0,6)$

$\vec{r}_{vv} = (0,0,2u^3)$ $\quad$ $\vec{r}_{vv}(1,1) = (0,0,2)$

$E = 10$ $\quad$ $L = \frac{6}{\sqrt{14}}$

$F = 6$ $\quad$ $M = \frac{6}{\sqrt{14}}$

$G = 5$ $\quad$ $N = \frac{2}{\sqrt{14}}$

$$K = \frac{LN-M^2}{EG-F^2} = \frac{\frac{12}{14} - \frac{36}{14}}{14} = -\frac{24}{14^2} = -\frac{6}{49} < 0 \Rightarrow \vec{r}(1,1) \text{ je hiperbolična točka}$$

$$H = \frac{EN-GL-2MF}{2(EG-F^2)} = \frac{\frac{20}{\sqrt{14}} + \frac{30}{\sqrt{14}} - \frac{72}{\sqrt{14}}}{2\cdot 14} = -\frac{11}{14\sqrt{14}}$$