# DODATNE NALOGE

1. S POMOČJO RIEMANNOVIH VSOT IZRAČUNAJ

(a) $$\lim_{n\to\infty} \frac{1^s + 2^s + \dots + n^s}{n^{s+1}}$$

(b) NAJ BO $f: [0,1] \to (0,\infty)$ ZVEZNA. POENOSTAVI

$$\lim_{n\to\infty} \sqrt[n]{f(\tfrac{1}{n})\cdot f(\tfrac{2}{n}) \cdots f(\tfrac{n}{n})}$$

2. IZRAČUNAJ $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \cos(nx)\, dx$, $n \in \mathbb{N}$.

3. (a) NAJ BO $f: [0,1] \to \mathbb{R}$ ZVEZNA. DOKAŽI

$$\int_0^{\pi} x f(\sin x)\, dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\, dx = \pi \int_0^{\frac{\pi}{2}} f(\sin x)\, dx$$

(b) IZRAČUNAJ INTEGRAL

$$\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx$$

REŠITVE:

1 (a)

$$L=\lim_{n\to\infty}\frac{1^s+2^s+\dots+n^s}{n^{s+1}}=$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[\left(\frac{1}{n}\right)^s+\left(\frac{2}{n}\right)^s+\dots+\left(\frac{n}{n}\right)^s\right]=$$

$$\underset{\text{R-VSOTA}}{=}\int_0^1 x^s\,dx=\frac{x^{s+1}}{s+1}\Big|_0^1=\frac{1}{s+1}$$

(b) $L=\lim_{n\to\infty}\sqrt[n]{f(\tfrac{1}{n})\cdot\ldots\cdot f(\tfrac{n}{n})}$

$$\ln L=\ln\lim_{n\to\infty}\sqrt[n]{f(\tfrac{1}{n})\cdot\ldots\cdot f(\tfrac{n}{n})}=\lim_{n\to\infty}\ln(\#)=$$

$$=\lim_{n\to\infty}\frac{1}{n}\left(\ln f(\tfrac{1}{n})+\dots+\ln f(\tfrac{n}{n})\right)=\int_0^1\ln f(x)\,dx$$

$$\Rightarrow L=e^{\int_0^1\ln f(x)\,dx}$$

② $$I_n = \int_0^{\pi/2} \cos^n x \cos(nx)\,dx$$

PER PARTES

$$u = \cos^n x \qquad dv = \cos(nx)\,dx$$

$$du = -n\cos^{n-1}x \sin x\,dx \qquad v = \frac{\sin(nx)}{n}$$

$$I_n = \underbrace{\cos^n x \,\frac{\sin(nx)}{x}\Big|_0^{\frac{\pi}{2}}}_{=\,0} + \int_0^{\frac{\pi}{2}} \cos^{n-1} x \sin x \sin(nx)\,dx$$

UPORABIMO:

$$\sin\alpha \cdot \sin\beta = \frac{1}{2}\big(\cos(\alpha-\beta) - \cos(\alpha+\beta)\big)$$

$$I_n = \underbrace{\frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos((n-1)x)\,dx}_{I_{n-1}} - \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos((n+1)x)\,dx$$

RAZCEPIMO ŠE:

$$\cos((n+1)x) = \cos(nx)\cos x - \sin(nx)\sin(x)$$

POSLEDICA:

$$\int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos(n+1)x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x \cos(nx)\,dx - \int_0^{\frac{\pi}{2}} \cos^{n-1} x \sin x \sin(nx)\,dx$$

$$= I_n - I_n = 0$$

TO SMO DOBILI PO PER PARTESU

OSNOVNA DEFINICIJA

SKLEP: $$I_n = \frac{1}{2} I_{n-1} = \frac{1}{2^{n-1}} \cdot I_1 = \frac{1}{2^{n-1}} \int_0^{\pi/2} \cos^2 x\,dx = \frac{1}{2^{n-1}} \cdot \frac{\pi}{4}$$

③ (a)

$$\int_0^{\pi} x\, f(\sin x)\,dx \overset{\#2}{=} \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx \overset{\#1}{=} \pi\int_0^{\frac{\pi}{2}} f(\sin x)\,dx$$

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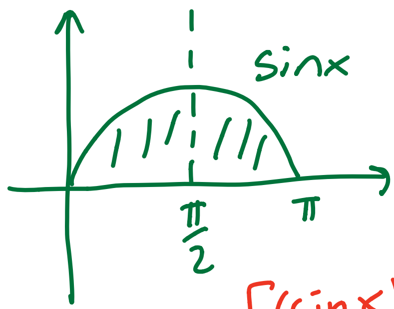
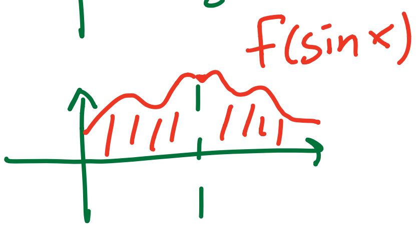
DOKAŽIMO #1:




SINUS JE SIMETRIČEN NA $x=\frac{\pi}{2}$.
ZATO JE TAK TUDI KOMPOZITUM.

t.j. $\displaystyle\int_0^{\pi} f(\sin x)\,dx = 2\int_0^{\frac{\pi}{2}} f(\sin x)\,dx$

TO ZADOŠČA.

DOKAŽIMO #2:

$$x=\pi-t \Rightarrow dx=-dt$$

$$\int_0^{\pi} x\, f(\sin x)\,dx = \int_{\pi}^{0} (\pi-t)\, f(\sin(\pi-t))(-dt) =$$

$$= \pi\int_0^{\pi} f(\sin t)\,dt - \int_0^{\pi} t\cdot f(\sin(t))\,dt$$

TO JE 2x DESNA STRAN

TA INTEGRALA STA ENAKA

(b) UPORABIMO TO ZVEZO:

$f(x)=\frac{x}{2-x^2}$ ZA #2

$$\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^{\pi}\frac{x\sin x}{2-\sin^2 x}\,dx = \frac{\pi}{2}\int_0^{\pi}\frac{\sin x}{2-\sin^2 x} =$$

$$= \frac{\pi}{2}\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}\,dx = -\frac{\pi}{2}\int_1^{-1}\frac{dt}{1+t^2} = \frac{\pi}{2}\left(\operatorname{arctg}1-\operatorname{arctg}(-1)\right)$$

$$= \frac{\pi}{2}\cdot\frac{\pi}{2} = \frac{\pi^2}{4}$$

$t=\cos x$
$dt=-\sin x\,dx$

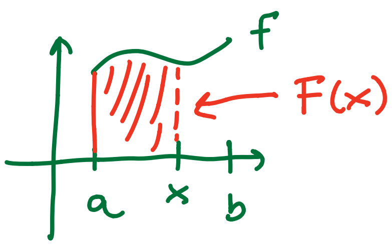
NALOGA: $0<a<b$, $f:[a,b]\to(0,\infty)$ ZVEZNA.

$$F(x)=\int_a^x f(t)\,dt\ ,\quad G(x)=\int_a^x t\cdot f(t)\,dt$$

DOKAŽI, DA JE $F/G$ DOBRO DEFINIRANA IN PADAJOČA NA $(a,b)$.

REŠITEV:

KAJ PREDSTAVLJA $F$?




PRIMITIVNO FUNKCIJO t.j. FUNKCIJO, KI $x$ PRIREDI PLOŠČINO POD $f$ NA $[a,x]$.

PODOBNO JE DEFINIRANA $G$ ZA $t\cdot f(t)$.
KER JE $f(t)>0$ IN $t>0$, JE $t\cdot f(t)>0$ IN $G(x)>0$ ZA VSE $x>a$ t.j. $G\neq 0$ IN $F/G$ JE DEFINIRANA.

OSNOVNI IZREK ANALIZE:

$$\left(\int_a^x f(t)\,dt\right)'=f(x)$$

ODVOD PRIMITIVNE FUNKCIJE JE ZAČETNA FUNKCIJA

V NAŠEM PRIMERU:

$$F'(x)=f(x)\ ,\quad G'(x)=x\,f(x)$$

$$\left(\frac{F}{G}\right)'=\frac{F'G-FG'}{G^2}=\frac{1}{G^2}\Big(f(x)\cdot G-F(x)\cdot x\cdot f(x)\Big)=$$

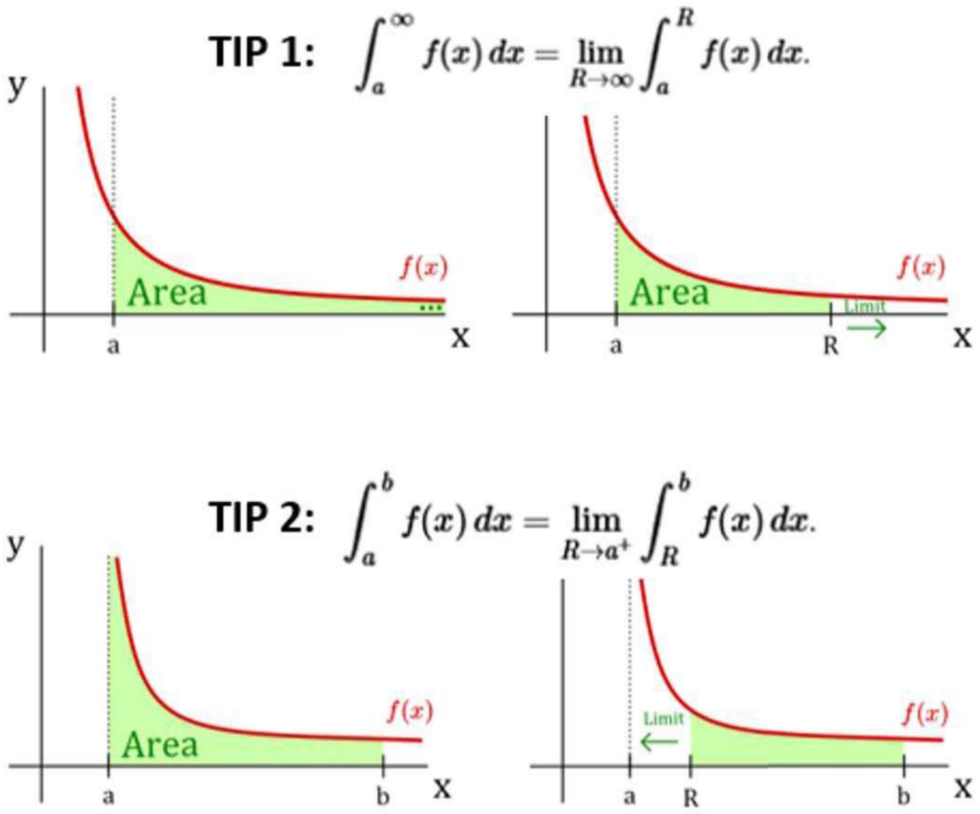
$$=\frac{f}{G^2}\left(\int_a^x\big(t\,f(t)-x\,f(t)\big)\,dt\right)=$$

$$=\frac{f}{G^2}\left(\int_a^x(t-x)f(t)\,dt\right)<0,$$ KER $f>0$, $G^2>0$, $(t-x)\leq 0$ ZA $t\in[a,x]$.

# Posplošeni integral

Uros KUZMAN

Obravnavali bomo dva tipa izlimitiranih oz. posplošenih integralov.




V praksi to pomeni

$$\int_a^\infty f(x)dx = \lim_{x\to\infty} F(x) - F(a), \quad \int_a^b f(x)dx = F(b) - \lim_{x\searrow a} F(x).$$

Na enak način obravnavamo tudi $-\infty$ in pol v zgornji meji.

Primer:

$$I_b = \int_0^b \frac{dx}{\sqrt{|x-1|}}, \; b \in \{1, 3, \infty\} .$$

Prvi intergal ima težavo le v zgornji meji

$$I_1 = \int_0^1 \frac{dx}{\sqrt{1-x}} = \left[-2(1-x)^{\frac{1}{2}}\right]_0^1 = 0 + 2 = 2.$$

Drugi integral ima pol sredi območja zato ga razbijemo na dva dela

$$I_3 = \int_0^1 \frac{dx}{\sqrt{1-x}} + \int_1^3 \frac{dx}{\sqrt{x-1}}.$$

Prvi del smo že izračunali, zato imamo

$$I_3 = 2 + \left[2(x-1)^{\frac{1}{2}}\right]_1^3 = 2 + 2\sqrt{2} - 0 = 2 + 2\sqrt{2}.$$

Pri tretjem integralu postopamo podobno

$$I_\infty = 2 + \int_1^\infty \frac{dx}{\sqrt{x-1}} = 2 + \left[2(x-1)^{\frac{1}{2}}\right]_1^\infty = 2 + 2 \lim_{x\to\infty} \sqrt{x-1}.$$

Ker limita ne konvergira, je nekonvergenten tudi integral.

Primer:

$$I_a = \int_a^\infty \frac{dx}{x\ln^2 x}, \quad a \in \{e, 1\}.$$

Vpeljimo spremenljivko $t = \ln x$ oz. $dt = \frac{dx}{x}$. Dobimo

$$I_a = \int_{\ln a}^\infty \frac{dt}{t^2} = \left[-\frac{1}{t}\right]_{\ln a}^\infty.$$

V primeru, ko je $a = e$, dobimo

$$I_e = -\lim_{t\to\infty}\frac{1}{t} + \frac{1}{\ln e} = 1.$$

V primeru, ko je $a = 1$, dobimo

$$I_1 = -\lim_{t\to\infty}\frac{1}{t} + \lim_{t\to 0}\frac{1}{t}.$$

Ta integral ni konvergenten, ker druga limita ne obstaja.

Primer:

$$I = \int_0^1 \ln x dx.$$

Ta integral je posplošen zaradi logaritemskega pola pri $x = 0$. Rešimo ga z intgracijo po delih

$$u = \ln x \Rightarrow du = \frac{dx}{x},$$

$$dv = dx \Rightarrow v = x.$$

Dobimo

$$I = [x \ln x]_0^1 - \int_0^1 dx = -\lim_{x \searrow 0} x \ln x - 1.$$

To limito izračunamo z L'Hospitalovim pravilom

$$\lim_{x \searrow 0} x \ln x = \lim_{x \searrow 0} \frac{\ln x}{x^{-1}} = \lim_{x \searrow 0} \frac{x^{-1}}{-x^{-2}} = -\lim_{x \searrow 0} x = 0 \Longrightarrow I = -1.$$

Primer:

$$I_n = \int_0^\infty x^n e^{-x},\ n \in \mathbb{N}.$$

Uporabimo integracijo po delih za $u = x^n$ in $dv = e^{-x}dx$. Imamo

$$du = nx^{n-1},\ v = -e^{-x}$$

Torej je

$$I_n = \left[-x^n e^{-x}\right]_0^\infty + n\int_0^\infty x^{n-1}e^{-x}dx = 0 + nI_{n-1}.$$

Sedaj postopamo induktivno

$$I_n = nI_{n-1} = \ldots = n!\, I_0.$$

Nazadje izračunamo še

$$I_0 = \int_0^\infty e^{-x}dx = \left[-e^{-x}\right]_0^\infty = 1 \Longrightarrow I_n = n!.$$

Primer: Za katere polinome $p(x)$ stopnje 2 obstaja integral

$$I = \int_0^1 \frac{p(x)dx}{x^3 - 3x + 2}?$$

Velja:

$$x^3 - 3x + 2 = (x-1)^2(x+2),$$

kar pomeni, da ima integrand pol pri $x = 1$. Nedoločeni integral te racionalne funkcije je enak

$$\int \frac{p(x)dx}{x^3 - 3x + 2} = A\ln|x-1| + B\ln|x+2| + \frac{C}{x-1} + D.$$

Če želimo, da je določeni integral mogoče izračunati v $x = 1$, morata biti $A = C = 0$. To pomeni, da mora biti

$$\frac{p(x)}{x^3 - 3x + 2} = \frac{k}{x+2} \Rightarrow p(x) = k(x-1)^2,\ k \in \mathbb{R}.$$

V tem primeru je $I = k(\ln 3 - \ln 2)$.