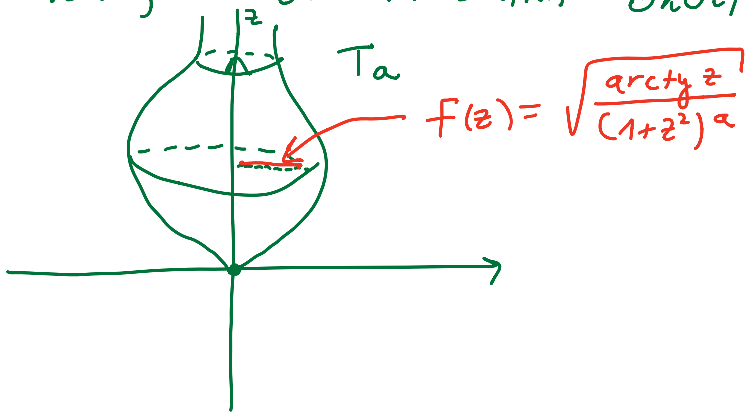
1. NAJ BO $T_a = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leq \frac{\operatorname{arctg} z}{(1+z^2)^a}\}$, $a>0$.

(a) ZA KATERE a JE PROSTORNINA T_a KONČNA?

(b) IZRAČUNAJ PROSTORNINO T_1.

REŠITEV:

POSKUSIMO SKICIRATI TELO T_a. VEMO, DA SO NJEGOVI PRESEKI Z RAVNINAMI $z=$ konst. ZAPRTI KROGI S POLMEROM $\sqrt{\frac{\operatorname{arctg} z}{(1+z^2)^a}}$. TOREJ GRE ZA TELO, KI JE VRTENINA OKOLI OSI z.



(a) KER GRE ZA VRTENINO OKOLI OSI z, JE VOLUMEN:

$$V_a = \pi \int_0^\infty f(z)^2 \, dz = \pi \int_0^\infty \frac{\operatorname{arctg} z}{(1+z^2)^a} \, dz$$

TA INTEGRAL IMA SINGULARNOST PRI $z=\infty$. TAM PIŠEMO:

$$\frac{\operatorname{arctg} z}{(1+z^2)^a} = \frac{\frac{\operatorname{arctg} z}{(\frac{1}{z^2}+1)^a}}{z^{2a}} = \frac{g(z)}{z^{2a}}, \quad \lim_{z\to\infty} g(z) = \frac{\pi}{2}, \quad s = 2a.$$

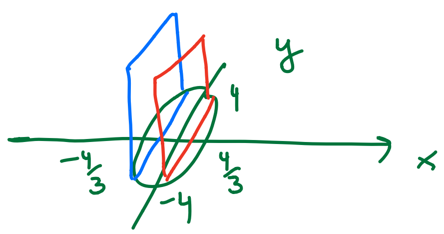
SKLEP: V_a OBSTAJA ZA $a > \frac{1}{2}$.

(b) $$V_1 = \pi \int_0^\infty \frac{\operatorname{arctg} z}{1+z^2} \, dz = \pi \int_0^{\frac{\pi}{2}} t \, dt = \pi \cdot \frac{\pi^2}{8} = \frac{\pi^3}{8}$$

$$t = \operatorname{arctg} z$$
$$dt = \frac{dz}{1+z^2}$$

② TELO JE NAVZDOL OMEJENO Z RAVNINO $z=0$, NJEGOVA PROJEKCIJA NA TO RAVNINO JE ELIPSA $9x^2+y^2=16$, NJEGOVI PRESEKI Z RAVNINAMI $x=c$ PA SO KVADRATI. IZRAČUNAJ VOLUMEN.

REŠITEV: ZAČNIMO S SKICO.



$$9x^2+y^2=16 \;/:16$$

$$\frac{x^2}{\frac{16}{9}}+\frac{y^2}{16}=1$$

$$a=\frac{4}{3} \qquad b=4$$

SEŠTETI MORAMO KONTINUUM MNOGO KVADROV S STRANICAMI $2y, 2y, dx$. MED TEM, KO NAM x TEČE OD $-\frac{4}{3}$ DO $\frac{4}{3}$,

t.j. 'ENA REZINA' IMA VOLUMEN:

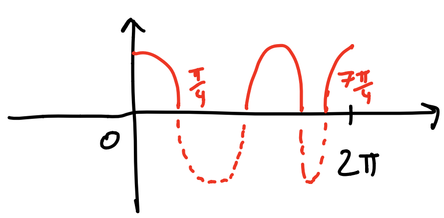
$$dV=(2y)^2\,dx=4y^2\,dx=4(16-9x^2)\,dx$$

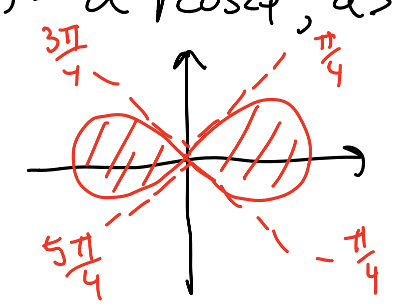
CEL VOLUMEN JE TOREJ:

$$V=\int_{-\frac{4}{3}}^{\frac{4}{3}}4(16-9x^2)\,dx=2\int_0^{\frac{4}{3}}4(16-9x^2)\,dx=$$

$$=8\left(16x-3x^3\right)\Big|_0^{\frac{4}{3}}=\frac{1024}{9}$$

③ IZRAČUNAJ PLOŠČINO LEMNISKATE, KI JE V POLARNIH KOORDINATAH PODANA Z $r(\varphi)=a\sqrt{\cos 2\varphi}$, $a>0$.





REŠITEV:

$$S=\frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^2(\varphi)\,d\varphi+\frac{1}{2}\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} r^2(\varphi)\,d\varphi=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^2(\varphi)\,d\varphi=$$

$$=a^2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\cos 2\varphi\,d\varphi=\frac{a^2}{2}\sin 2\varphi\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}=\underline{\underline{a^2}}$$

(4) IZRAČUNAJ PLOŠČINO ZANKE PRI DESCARTESOVEM LISTU: $x(t)=\frac{3at}{1+t^3}$, $y(t)=\frac{3at^2}{1+t^3}$, $a>0$.

REŠITEV: SKICIRAJMO KRIVULJO:

$x(t)$: NIČLA PRI $t=0$

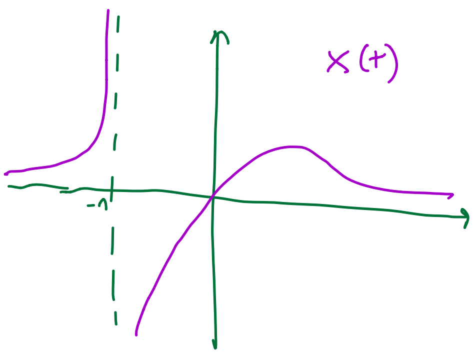
LIMITI $x\to 0$, KO $t\to\pm\infty$

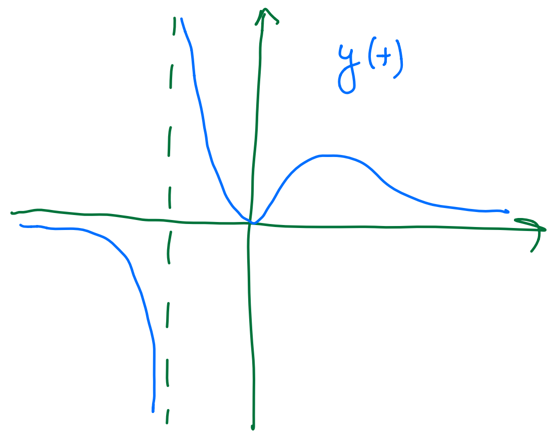
POL PRI $t=-1$

$y(t)$: DVOJNA NIČLA PRI $t=0$

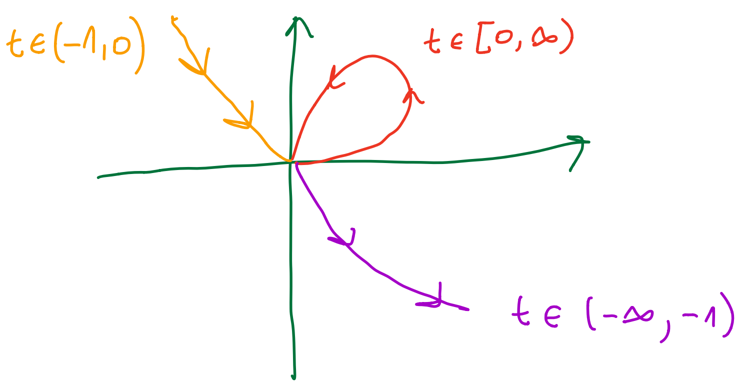
LIMITI $y\to 0$ ZA $t\to\pm\infty$

POL PRI $t=-1$





SKICA KRIVULJE:



$$S=\frac{1}{2}\int_0^\infty (x\dot{y}-\dot{x}y)\,dt$$

$$\dot{x}=\frac{3a(1-2t^3)}{(1+t^3)^2}$$

$$\dot{y}=\frac{3at(2-t^3)}{(1+t^3)^2}$$

$$x\dot{y}-\dot{x}y=\frac{9a^2t^2}{(1+t^3)^2}$$

$u=1+t^3$

$du=3t^2dt$

$$S=\frac{1}{2}\int_0^\infty\frac{9a^2t^2}{(1+t^3)^2}=\frac{1}{2}\int_1^\infty\frac{3a^2}{u^2}\,du=-\frac{3a^2}{2u}\Big|_1^\infty=\underline{\underline{\frac{3a^2}{2}}}$$