LINEARNA ALGEBRA 2020/21
24. VAJE: 14. 4. 2021

1. Poišči kakšno ortogonalno (ortonormirano bazo) prostora
$$V = \mathrm{Lin}\{(1,1,0,0)^T, (0,1,1,0)^T, (-1,1,1,1)^T\}.$$

2. Poišči ortogonalno bazo prostora $\mathbb{R}_3[x]$, glede na skalarni produkt $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)dt$.

3. Naj bo $\{v_1, \ldots, v_k\}$ ortonormirana množica (glede na nek skalarni produkt) v vektorskem prostoru V. Za poljuben $u \in V$ poišči $v \in \mathrm{Lin}\{v_1, \ldots, v_k\}$, ki mu je najbližje.

4. Naj bodo V_i podprostori vektorskega prostora s skalarnim produktom W. Pokaži,
   a.) Če je S podmnožica W, potem je $(S^{\perp})^{\perp} = \mathrm{lin}\, S$.
   b.) $V_1 \subset V_2 \implies V_2^{\perp} \subset V_1^{\perp}$
   c.) $(V_1 + V_2)^{\perp} = V_1^{\perp} \cap V_2^{\perp}$
   d.) $(V_1 \cap V_2)^{\perp} = V_1^{\perp} + V_2^{\perp}$

5. Poišči ortogonalni komplement
$$\mathrm{lin}\left\{\begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \\ 0 \\ -1 \end{bmatrix}.\right\}$$
   v $\mathbb{R}^5$ glede na standardni skalarni produkt.

1. Poišči kakšno ortogonalno (ortonormirano bazo) prostora

$$V = \mathrm{Lin}\{(1,1,0,0)^T, (0,1,1,0)^T, (-1,1,1,1)^T\}.$$

Gram-Schmidtova ortogonalizacija

V - vektorski prostor s skalarnim produktom

Vhodni podatki $v_1, v_2, \ldots, v_k \in V$

Izhod $\{u_1, u_2, \ldots, u_k\}$ - ortogonalna baza $\mathrm{Lin}\{v_1, \ldots, v_k\}$

$$u_1 = v_1$$

$$u_2 = v_2 - \frac{\langle v_2, u_1\rangle}{\langle u_1, u_1\rangle} u_1$$

$$u_3 = v_3 - \frac{\langle v_3, u_1\rangle}{\langle u_1, u_1\rangle} u_1 - \frac{\langle v_3, u_2\rangle}{\langle u_2, u_2\rangle} u_2$$

$$\vdots$$

$$u_j = v_j - \sum_{l<j} \frac{\langle v_j, u_l\rangle}{\langle u_l, u_l\rangle} u_l$$

$$\vdots$$

Če je $u_j = 0$, ga izpustimo.

Če na vsakem koraku še normaliziramo $u_j \to \frac{u_j}{\|u_j\|}$ na koncu dobimo ortonormirano množico.

Ortogonaliziraj glede na standardni skalarni produkt

$$\boxed{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}} \quad \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}}$$

$$u_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{0}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \underset{\frac{2}{3/2}}{\frac{4}{3}}\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix}}$$

Za ortonormiranje $\|u_1\| = \sqrt{2}$ $\quad \|u_2\| = \sqrt{\frac{3}{2}}$ $\quad \|u_3\| = \sqrt{\frac{4}{3}}$

2. Poišči ortogonalno bazo prostora $\mathbb{R}_3[x]$, glede na skalarni produkt $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)dt$.

skalarni produkt na prostoru zveznih funkcij $[-1,1] \to \mathbb{R}$

$1, x, x^2, x^3$

$\boxed{1}$

$$\langle 1, 1\rangle = \int_{-1}^{1} 1\, dt = 2$$

$\boxed{x}$

$$\langle x, 1\rangle = \int_{-1}^{1} t\, dt = 0$$

$$\langle x^2, 1\rangle = \int_{-1}^{1} t^2\, dt = \left(\frac{1}{3}t^3\right)\Big|_{-1}^{1} = \frac{2}{3}$$

$$x^2 - \frac{\frac{2}{3}}{2} 1$$

$\boxed{x^2 - \frac{1}{3}}$

$$\langle x^2, x\rangle = \int_{-1}^{1} t^3\, dt \cancel{= 0}$$

$$x^3 - \frac{\frac{2}{5}}{\frac{2}{3}} x$$

$$\langle 1, x^3\rangle = \int_{-1}^{1} t^3\, dt = 0$$

$$\langle x, x^3\rangle = \int_{-1}^{1} t^4\, dt = \left(\frac{1}{5}t^5\right)\Big|_{-1}^{1} = \frac{2}{5}$$

$\boxed{x^3 - \frac{5}{3}x}$

$$\langle x^2 - \frac{1}{3}, x^3\rangle = \int_{-1}^{1} t^5 - \frac{1}{3}t^3\, dt = 0$$

$$\langle x, x\rangle = \frac{2}{3}$$

3. Naj bo $\{v_1, \ldots, v_k\}$ ortonormirana množica (glede na nek skalarni produkt) v vektorskem prostoru V. Za poljuben $u \in V$ poišči $v \in \text{Lin}\{v_1, \ldots, v_k\}$, ki mu je najbližje.

$d(x,y) = \|x-y\| = \langle x-y, x-y \rangle$ – raz je merimo z normo.

Z $u \in V$ iščemo $v \in \text{lin}\{v_1, \ldots, v_k\}$ za katerega je $\|u-v\|$ najmanjša. Iščemo tak v, da bo $u - v \perp \text{lin}\{v_1, \ldots, v_k\}$.

Kratek dokaz: Naj bo u_p pravokotna projekcija u na $\text{lin}\{v_1, \ldots, v_k\}$. Vsak $v \in \text{lin}\{v_1, \ldots, v_k\}$ lahko zapišemo kot $u_p + v_p = v$ z enolično določenim $v_p \in \text{lin}\{v_1, \ldots, v_k\}$

$$\langle (u-u_p) - v_p, (u-u_p) - v_p \rangle = \langle u - u_p, u - u_p \rangle - \langle u - u_p, v_p \rangle - \langle v_p, u_p - v_p \rangle + \langle v_p, v_p \rangle$$

$$= \|u-u_p\|^2 + \|v_p\|^2 \overset{\text{enakost je dosežena, ko je } v_p = 0}{\geq} \|u-u_p\|^2.$$

Za pravokotno projekcijo rešujemo enačbe

$$j = 1, \ldots, k \qquad \left\langle u - \sum_{i=1}^{k} \alpha_i v_i, v_j \right\rangle = 0$$

$$\| \quad \text{ostali sk. produkti so enaki } 0$$

$$\langle u, v_j \rangle - \alpha_j \langle v_j, v_j \rangle = \langle u - v_j \rangle - \alpha_j$$

$$j = 1 \ldots k \qquad \alpha_j = \langle u, v_j \rangle \longrightarrow \boxed{\text{Projekcija } u_p = \sum_{j=1}^{k} \langle u, v_j \rangle v_j}$$

če v_j niso normirani dobimo $u_p = \sum_{j=1}^{k} \frac{\langle u, v_j \rangle}{\langle v_j, v_j \rangle} v_j$

Primer „praktične" uporabe

V - prostor zveznih funkcij $[a,b] \to \mathbb{R}$. Za funkcijo f iščemo polinom p_f stopnje največ n, ki je najbližje f glede na $\langle q_1, q_2\rangle = \int_{-1}^{1} q_1 q_2 \, dt$

• Izračunamo pravokotno projekcijo f na $\mathbb{R}_n[x]$.

4. Naj bodo V_i podprostori vektorskega prostora s skalarnim produktom W. Pokaži,
   a.) Če je S podmnožica W, potem je $(S^\perp)^\perp = \operatorname{lin} S$.
   b.) $V_1 \subset V_2 \implies V_2^\perp \subset V_1^\perp$
   c.) $(V_1 + V_2)^\perp = V_1^\perp \cap V_2^\perp$
   d.) $(V_1 \cap V_2)^\perp = V_1^\perp + V_2^\perp$

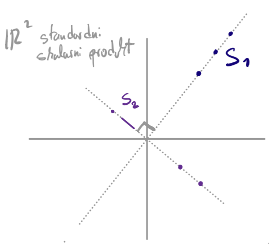
Ortogonalni komplement

$S_1, S_2 \subseteq W$ — vektorski prostor s sk. prod.

poljubni podmnožici

S_1 in S_2 sta si ortogonalni $S_1 \perp S_2$, če je $(s_1, s_2) = 0 \ \forall s_1 \in S_1, s_2 \in S_2$

Primer:

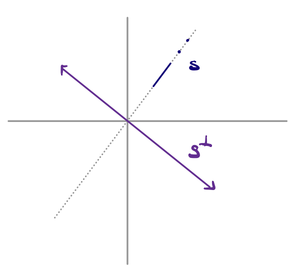


$S_1 \perp S_2$

ortogonalni komplement

$S \subseteq W \qquad S^\perp = \{ w \in W \mid \langle w, s\rangle = 0 \ \forall s \in S \}$

Primer:



(Iz predavanj)

- za vsak $S \subseteq W$, je $S^\perp = \text{lin}\, S^\perp$ vektorski podprostor.
- če je $V \leq W$ vektorski podprostor je $(V^\perp)^\perp = V$ in $W = V \oplus V^\perp$

a) $(S^\perp)^\perp = \text{lin}\, S$

- $S^\perp = (\text{lin}\, S)^\perp \rightarrow (S^\perp)^\perp = (\text{lin}\, S)^{\perp\perp} = \text{lin}\, S.$

b) $V_1 \subseteq V_2 \Rightarrow V_2^\perp \subseteq V_1^\perp$

Naj bo $x \in V_2^\perp$, želimo pokazati, da je $x \in V_1^\perp$

$$\langle x, v_2 \rangle = 0 \ \forall\, v_2 \in V_2 \Longrightarrow \langle x, v_1 \rangle = 0 \ \forall\, v_1 \in V_1$$

$(v_1 \in V_1 \Rightarrow v_1 \in V_2)$

c.) $(V_1 + V_2)^\perp = V_1^\perp \cap V_2^\perp$ („De Morganov zakon")

$x \in (V_1 + V_2)^\perp \Leftrightarrow \forall\, v_1 \in V_1,\ v_2 \in V_2 \ \langle x, v_1 + v_2 \rangle = 0$

$\Updownarrow$

$\forall v_1 \ \langle x, v_1 \rangle = 0$ in $\forall \ \langle x, v_2 \rangle = 0$

$\forall v_1 \in V_1 \ v_2 \in V_2 \quad \langle x, v_1 \rangle + \langle x, v_2 \rangle = 0$

izberemo $(v_1, 0)$, $(0, v_2) \rightarrow \forall v_1 \in V_1 \ \langle x, v_1 \rangle = 0$, $\forall v_2 \in V_2 \ \langle x, v_2 \rangle = 0$

Če je $\langle x, v_1 \rangle = 0$ in $\langle x, v_2 \rangle = 0$ je tudi $\langle x, v_1 + v_2 \rangle = 0.$

$x \in V_1^\perp \cap V_2^\perp \Leftrightarrow x \in V_1^\perp$ in $x \in V_2^\perp$

d) $(V_1 \cap V_2)^\perp = V_1^\perp + V_2^\perp$ (Tu drugi „Demorganov zakon")

(Lahko pokažemo tudi direktno)

Uporabimo c) $(U_1 + U_2)^\perp = U_1^\perp \cap U_2^\perp$

$$\Downarrow$$

$$U_1 + U_2 = (U_1^\perp \cap U_2^\perp)^\perp$$

Vstavimo $U_1 = V_1^\perp \quad U_2 = V_2^\perp$

$$V_1^\perp + V_2^\perp = \left((V_1^\perp)^\perp \cap (V_2^\perp)^\perp\right)^\perp = (V_1 \cap V_2)^\perp$$

5. Poišči ortogonalni komplement

$$V = \mathrm{lin}\left\{\begin{bmatrix}1\\-2\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}3\\-1\\2\\0\\-1\end{bmatrix}.\right\}$$

v $\mathbb{R}^5$ glede na standardni skalarni produkt.

Postopek 1

Poiščemo bazo prostora V $B = \{v_1, \dots, v_k\}$. Bazo dopolnimo do baze celotnega prostora $B' = B \cup \{u_1, \dots, u_l\}$.

Na B' uporabimo GS postopek. (Vrstni red: najprej v_i-ji potem u_i-ji)

Prvih k-vrjenih so ortogonalna baza V, preostali so ortogonalna baza $V^\perp$

Postopek 2 Izberemo množico, ki razpenja V $\{v_1,\dots,v_m\}$ in množico dopolnimo, do množice $\{v_1,\dots,v_m,u_1,\dots,u_l\}$, ki razpenja celoten prostor. Na $\{v_1,\dots,v_m,u_1,\dots,u_l\}$ uporabimo GS. Neničelni vektorji izmed prvih m-vrnjenih so baza V preostali neničelni so baza $V^\perp$

$$\boxed{\underset{v_1}{\begin{bmatrix}1\\-2\\0\\1\\0\end{bmatrix}}},\ \underset{v_2}{\begin{bmatrix}0\\1\\0\\1\\1\end{bmatrix}}\ \underset{v_3}{\begin{bmatrix}3\\-1\\2\\0\\-1\end{bmatrix}},\ \underset{u_1}{\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}},\ \underset{u_2}{\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}}$$

$$v_1 = x_1$$

$$\begin{bmatrix}0\\1\\0\\1\\1\end{bmatrix}+\frac{1}{6}\begin{bmatrix}1\\-2\\0\\1\\0\end{bmatrix}=\boxed{\begin{bmatrix}\frac{1}{6}\\\frac{2}{3}\\0\\\frac{7}{6}\\1\end{bmatrix}} = x_2$$

$$\langle x_1, x_1\rangle = 6$$

$$\langle x_1, v_2\rangle = -1$$

$$\langle x_1, v_3\rangle = 5$$

$$\langle x_2, v_3\rangle = \frac{3}{6}-\frac{2}{3}-1=-\frac{7}{6}$$

$$\langle x_2, x_2\rangle = \frac{102}{36}=\frac{17}{6}$$

$$\begin{bmatrix}3\\-1\\2\\0\\-1\end{bmatrix}-\frac{5}{6}\begin{bmatrix}1\\-2\\0\\1\\0\end{bmatrix}+\frac{7}{17}\begin{bmatrix}\frac{1}{6}\\\frac{2}{3}\\0\\\frac{7}{6}\\1\end{bmatrix}=\boxed{\begin{bmatrix}\frac{38}{17}\\\frac{16}{17}\\2\\-\frac{6}{17}\\-\frac{10}{17}\end{bmatrix}}$$

...

Izračunamo (računalnik izračuna) ortonormirano bazo

$$\begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\sqrt{\frac{2}{3}} \\ 0 \\ \frac{1}{\sqrt{6}} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{102}} \\ 2\sqrt{\frac{2}{51}} \\ 0 \\ \frac{7}{\sqrt{102}} \\ \sqrt{\frac{6}{17}} \end{bmatrix}, \begin{bmatrix} \frac{19}{2\sqrt{187}} \\ \frac{4}{\sqrt{187}} \\ \frac{1}{2}\sqrt{\frac{17}{11}} \\ -\frac{3}{2\sqrt{187}} \\ -\frac{5}{2\sqrt{187}} \end{bmatrix}, \boxed{\begin{bmatrix} -\frac{1}{2}\sqrt{\frac{15}{11}} \\ \frac{2}{\sqrt{165}} \\ -\frac{19}{2\sqrt{165}} \\ -\frac{7}{2\sqrt{165}} \\ \frac{1}{2}\sqrt{\frac{3}{55}} \end{bmatrix}}$$

$$\boxed{\begin{bmatrix} 0 \\ \frac{1}{\sqrt{15}} \\ -\frac{1}{\sqrt{15}} \\ \frac{2}{\sqrt{15}} \\ -\sqrt{\frac{3}{5}} \end{bmatrix}}$$

Baza ortogonalnega komplementa!

Linearni funkcionali

V - vektorski prostor nad $(\mathbb{R}$ ali $\mathbb{C}) = \mathbb{F}$

Linearno preslikavo $f: V \to \mathbb{F}$ imenujemo linearen funkcional

Če ima V skalarni produkt potem je za vsak $v \in V$ $f_v(x) = \langle x, v \rangle$ linearen funkcional

Rieszov izrek o reprezentaciji
Naj bo V vektorski prostor in $f: V \to \mathbb{F}$ lin. funkcional
Potem obstaja enolično določen $v \in V$, da je $\forall x \in V: f(x) = \langle x, v \rangle$.

Poišči polinom, ki reprezentira lin. funkcional

$V = \mathbb{R}_2[x]$ $\quad Lp = p(0)$, glede na skalarna produkta

$\langle f, g \rangle_1 = \int_{-1}^{1} fg\,dt$ $\quad \boxed{\# \; \langle f, g \rangle_2 = \int_0^1 f(t)g(t)\,dt}$

\# Iščemo $q \in \mathbb{R}_2[x]$ $\quad Lp = \langle p, q \rangle$

$1, x, x^2$ $\quad q = \alpha_0 + \alpha_1 x + \alpha_2 x^2$ iščemo $\alpha_0, \alpha_1, \alpha_2$

$\forall p \in \mathbb{R}_2[x] \qquad \langle p, q \rangle = p(0)$

Pogoj za bazne vektorje

$$\langle 1, q \rangle = 1 = L_1$$

$$\langle x, q \rangle = 0 = L_x$$

$$\underline{\langle x^2, q \rangle = 0} = L_{x^2}$$

$$\int_0^1 (\alpha_0 + \alpha_1 t + \alpha_2 t^2)\, dt = 1$$

$$\alpha_0 t + \frac{\alpha_1}{2} t^2 + \frac{\alpha_2}{3} t^3 \Big|_0^1 = \alpha_0 + \frac{\alpha_1}{2} + \frac{\alpha_2}{3} = 1$$

$$\int_0^1 (\alpha_0 t + \alpha_1 t^2 + \alpha_2 t^3)\, dt = \frac{\alpha_0}{2} + \frac{\alpha_1}{3} + \frac{\alpha_2}{4} = 0$$

$$\int_0^1 (\alpha_0 t^2 + \alpha_1 t^3 + \alpha_2 t^4)\, dt = \frac{\alpha_0}{3} + \frac{\alpha_1}{4} + \frac{\alpha_2}{5} = 0$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{3} & 1 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{3} & 1 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} \\ 0 & \frac{1}{12} & \frac{4}{45} & -\frac{1}{3} \end{array}\right]$$

$$\begin{array}{c} \\ 12 \\ 180 \end{array} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{3} & 1 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{180} & \frac{1}{6} \end{array}\right] \rightsquigarrow \text{Rešitev} \begin{bmatrix} 9 \\ -36 \\ 30 \end{bmatrix}$$

$\rightsquigarrow$ Iskan polinom je $\boxed{q = 9 - 36x + 30x^2}$.