LINEARNA ALGEBRA 2020/21
25. VAJE: 21. 4. 2021

1. V $\mathbb{R}_2[x]$ poišči vektorja, ki po Rieszovem izreku pripadata linearnemu funkcionalu $Lp = p(0)$, glede na skalarna produkta

$$\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)dt \quad \text{in} \quad \langle f, g \rangle = \int_{0}^{1} f(t)g(t)dt.$$

2. Naj bo $\mathbb{R}^3$ opremljen z običajnim skalarnim produktom. Izračunaj adjungirano preslikavo preslikave $x \mapsto a \times x$.

3. Naj bo $a \in U$ in $b \in V$, kjer sta U in V vektorska prostora s skalarnim produktom. Izračunaj adjungirano preslikavo linearne preslikave $U \to V$ določene z $x \mapsto \langle x, a \rangle b$.

   Pokaži, da je vsaka preslikava $A\colon U \to V$ z rang $A \leq 1$ oblike $Ax = \langle x, a \rangle b$ za neka $a \in U$ in $b \in B$.

4. Izračunaj adjungirano preslikavo odvajanju v $\mathbb{R}_2[x]$ glede na skalarni produkt $\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)dt$.

Slaba navodila

5. Naj bo V podprostor $\mathbb{R}_n[x]$ polinomov za katere velja $p(a) = p(b) = 0$ Izračunaj adjungirano preslikavo odvajanju $D\colon V \to \mathbb{R}_n[x]$ glede na skalarni produkt $\langle f, g \rangle = \int_a^b f(t)g(t)dt$.

6. Pokaži, da je $\ker(A^*A + B^*B) = \ker(A) \cap \ker(B)$ $(\ker(\sum_{j=1}^{n})A_j) = \bigcap_{j=1}^{n} \ker A_j)$

7. Naj bo $A\colon V \to V$ in W invarianten za A. Pokaži, da je $W^\perp$ invarianten za A^*.

1. V $\mathbb{R}_2[x]$ poišči vektorja, ki po Rieszovem izreku pripadata linearnemu funkcionalu $Lp = p(0)$, glede na skalarna produkta

Prejšji teden

$$\text{\textcircled{*}}\quad \langle f, g\rangle = \int_{-1}^{1} f(t)g(t)dt \quad \text{in} \quad \langle f, g\rangle = \int_{0}^{1} f(t)g(t)dt.$$

$L: V \to \mathbb{F}$ lin. funkcional

$\{b_1, \ldots, b_n\}$ baza V.

Iščemo $v \in V$, da bo $L(x) = \langle x, v\rangle$

v razvijemo po bazi $v = \sum_{i=1}^{n} \alpha_i b_i$

$$L(x) = \langle x, v\rangle = \langle x, \sum_{i=1}^{n} \alpha_i b_i\rangle = \sum_{i=1}^{n} \overline{\alpha_i} \langle x, b_i\rangle$$

Za x vstavimo b_j $j = 1, \ldots, n$ $\rightsquigarrow$ dobimo sistem linearnih enačb (n-enačb, n-spremenljivk $(\overline{\alpha_i})$)

Če je $\{b_i \mid i = 1, \ldots, n\}$ ortogonalna baza dobimo

$L(b_j) = \sum_{i=1}^{n} \overline{\alpha_i} \langle b_j, b_i\rangle = \overline{\alpha_j} \langle b_j, b_j\rangle \rightsquigarrow \boxed{\overline{\alpha_j} = \frac{L(b_j)}{\langle b_j, b_j\rangle}}$ $\xrightarrow{\text{ortonormirana baza}}$ $\alpha_j = \overline{L(b_j)}$

*

$$\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\,dt$$

$1,\ x,\ x^2 - \frac{1}{3}$ $\longleftarrow$ ortogonalna baza (prejšnje vaje)

$L(1) = 1 \qquad Lx = 0 \qquad L(x^2 - \frac{1}{3}) = -\frac{1}{3}$

$\int_{-1}^{1} 1 \cdot 1\,dt = 2 \qquad \langle x, x\rangle = \int_{-1}^{1} t^2\,dt = \frac{2}{3}$

$\langle x^2 - \frac{1}{3}, x^2 - \frac{1}{3}\rangle = \int_{-1}^{1} t^4 - \frac{2}{3}t^2 + \frac{1}{9}\,dt = \left(\frac{1}{5}t^5 - \frac{2}{9}t^3 + \frac{1}{9}t\right)\Big|_{-1}^{1} =$

$$= \frac{2}{5} - \frac{4}{9} + \frac{2}{9} = \frac{2}{5} - \frac{2}{9} = \frac{18-10}{45} = \frac{8}{45}$$

$$g = \frac{1}{2} \cdot 1 + \frac{0}{\frac{2}{3}} x - \frac{\frac{1}{3}}{\frac{8}{45}} \left(x^2 - \frac{1}{3} \right)$$

$$g = -\frac{15}{8} x^2 + \frac{9}{8}$$

2. Naj bo $\mathbb{R}^3$ opremljen z običajnim skalarnim produktom. Izračunaj adjungirano preslikavo preslikave $x \mapsto a \times x$.

Adjungirane preslikave

U, V - vektorska prostora s skalarnim produktom

$A: U \to V$ linearna preslikava $\quad A^*: V \to U$ (adjungirana preslikava) velja

$$\langle Au, v \rangle_V = \langle u, A^* v \rangle_U \quad \text{za } \forall\, v \in V,\ u \in U$$

A^* obstaja in je enolično določena z A in skalarnima produktoma.

Lastnosti

- $(A+B)^* = A^* + B^*$
- $(\alpha A)^* = \bar{\alpha} A^*$
- $(AB)^* = B^* A^*$
- $0^* = 0$
- $I^* = I$

$A: \mathbb{R}^3 \to \mathbb{R}^3 \quad Ax = a \times x \quad A^*$ glede na standardni skalarni prod

$$\langle Ax, y \rangle = \langle x, A^* y \rangle \quad \forall x, y$$

$$\parallel$$

$$\langle a \times x, y \rangle = \langle x, ?\, y \rangle$$

$$\parallel$$

$$[a, x, y] = [y, a, x] = \langle y \times a, \underline{x} \rangle \overset{\text{simetričnost}}{=}$$

$$= \langle x, y \times a \rangle = \langle x, \underline{-a \times y} \rangle$$

$$= \langle x, -Ay \rangle \rightsquigarrow A^* = -A$$

Preslikave s to lastnostjo imenujemo anti-simetrične

3. Naj bo $a \in U$ in $b \in V$, kjer sta U in V vektorska prostora s skalarnim produktom. Izračunaj adjungirano preslikavo linearne preslikave $U \to V$ določene z $x \mapsto \langle x, a \rangle b$.

(*) Pokaži, da je vsaka preslikava $A\colon U \to V$ z rang $A \leq 1$ oblike $Ax = \langle x, a \rangle b$ za neka $a \in U$ in $b \in B$.

$$\langle Au, v \rangle_V = \langle \langle u, a \rangle_U b, v \rangle_V = \langle u, a \rangle_U \langle b, v \rangle_V =$$

$$\langle u, \overline{\langle b, v \rangle_V} a \rangle_U = \langle u, \langle v, b \rangle a \rangle \rightsquigarrow A^* x = \langle x, b \rangle a$$

(*) Če je $\operatorname{rang} A = 0 \Rightarrow A = 0 \Rightarrow A = \langle x, 0 \rangle 0.$

$\operatorname{Rang} A = 1 \Rightarrow \operatorname{im} A = \operatorname{lin}\{b\}$ za nek $b \overset{0}{\#} \in V$. $\rightsquigarrow$ Za poljuben x je Ax nek raztег vektorja b. $Ax = \alpha(x) b.$ $\alpha(x) \in \mathbb{F}$

$\alpha(x)$ je linearen funkcional

$$A(x+y) = \alpha(x+y) b \longrightarrow \alpha(x+y) = \alpha(x) + \alpha(y)$$

$$\|$$

$$Ax + Ay = \alpha(x) b + \alpha(y) b = (\alpha(x) + \alpha(y)) b$$

$$A(\beta x) = \alpha(\beta x) b \rightsquigarrow \alpha(\beta x) = \beta \alpha(x)$$

$$\|$$

$$\beta A(x) = \beta \alpha(x) b$$

Rieszov izrek $\rightsquigarrow \alpha(x) = \langle x, a \rangle$ za nek $a \in U$.

$$Ax = \langle x, a \rangle b$$

Matrika adjungirane preslikave

$A : \mathbb{C}^n \to \mathbb{C}^m$ → standardna skalarna produkta na $\mathbb{C}^n, \mathbb{C}^m$

matrika $\left[\quad\right]$ ($m \times n$)

hermitsko transponiranje: $A^H = \left[\quad\right]$ ($n \times m$)

$A = (a_{ij})_{i=1\ldots m,\ j=1\ldots n}$

$A^H = (\overline{a_{ij}})_{i=1\ldots n,\ j=1\ldots m}$ transponiranje + konjugiranje

$$\langle x, y \rangle = y^H x$$

matrični produkt

$x \in \mathbb{C}^n \quad y \in \mathbb{C}^m$

$$\langle Ax, y \rangle_m = y^H A x = (y^H A) x = \left((y^H A)^H\right)^H x = (A^H y)^H x = \langle x, A^H y \rangle \rightsquigarrow \boxed{A^* = A^H}$$

- Splošno $A : V \to U \quad B_1, B_2$ bazi V, U

$[A]_{B_1 B_2}$ matrika preslikave iščemo $[A^*]_{B_2 B_1}$

$B_1 = \{b_1, \ldots, b_n\}$
$B_2 = \{b_1', \ldots, b_m'\}$ $\rightsquigarrow \langle A b_i, b_j' \rangle = \langle b_i, A^* b_j' \rangle$

Dobimo $n \cdot m$ enačb, in $n \cdot m$ neznank, ki se skrivajo v A^*.

- Če sta B_1, B_2 ortonormirani bazi.

Pazljivo z indeksi: morda so narobe!

$$Ax = \sum_{j=1}^m \alpha_j(x)\, b_j' \qquad \langle Ax, b_i' \rangle = \sum_{j=1}^m \alpha_j(x) \langle b_j', b_i' \rangle \to \alpha_j(x) = \langle Ax, b_j' \rangle$$

$$A b_i = \sum_{j=1}^m \alpha_{ji}\, b_j' \rightsquigarrow \alpha_{ji} = \langle A b_i, b_j' \rangle \qquad [A]_{B_1 B_2} = (\alpha_{ij})_{j=1,\ldots,n;\ i=1,\ldots,m}$$

$$\langle b_i, A^* b_j \rangle = \alpha_{ji}$$

$$[A^*]_{B_2 B_1} = \left([A]_{B_1 B_2}\right)^H$$

→ V ortonormiranih bazah računamo enako kot v standardni bazi s standardnim skalarnim produktom.

4. Izračunaj adjungirano preslikavo odvajanju v $\mathbb{R}_2[x]$ glede na skalarni produkt $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)dt$.

- Zapišemo matriko odvajanja v ortonormirani bazi →

→ matrika adjungirane preslikave je hermitska transponiranka

$1 \quad , \quad x \quad , \quad x^2 - \frac{1}{3}$

↓ normiramo

$$\boxed{\sqrt{2} \qquad \sqrt{\frac{2}{3}} \qquad \sqrt{\frac{45}{8}} = \frac{3}{2}\sqrt{\frac{5}{2}}}$$ ← norme

↓

$\frac{\sqrt{2}}{2} \quad , \quad \sqrt{\frac{3}{2}}\, x \quad , \quad \frac{3}{2}\sqrt{\frac{5}{2}}\left(x^2 - \frac{1}{3}\right)$

↓ odvajamo

$0 \quad , \quad \sqrt{\frac{3}{2}} \quad , \quad 3\sqrt{\frac{5}{2}}\, x$

$\sqrt{\frac{3}{2}} = \alpha \cdot \frac{\sqrt{2}}{2} \Rightarrow \alpha = \sqrt{3}$

$3\sqrt{\frac{5}{2}}\, x = \beta \cdot \sqrt{\frac{3}{2}}\, x \rightsquigarrow 3 \cdot \sqrt{\frac{5}{3}} = \sqrt{15} = \beta$

$$[A] = \begin{bmatrix} 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{15} \\ 0 & 0 & 0 \end{bmatrix} \qquad [A^*] = \begin{bmatrix} 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & \sqrt{15} & 0 \end{bmatrix}$$

! Slaba navodila !

5. Naj bo V podprostor $\mathbb{R}_n[x]$ polinomov za katere velja $p(a) = p(b) = 0$ Izračunaj adjungirano preslikavo odvajanju $D\colon V \to \mathbb{R}_n[x]$ glede na skalarni produkt $\langle f, g\rangle = \int_a^b f(t)g(t)dt$.

- Naiven poizkus

Per partes

$$\langle Df, g\rangle = \int_a^b f'(t)g(t)dt = \underbrace{f(t)g(t)\Big|_a^b}_{=0} - \int_a^b f(t)g'(t)dt$$

$f(a)=f(b)=0$

$$= \int_a^b f(t)\left(-g'(t)\right)dt$$

$D^* = -D$ problem! $-g'$ ni nujno element V.

„Popravljena pravila"

V - podprostor ∞ - zvezno odvedljivih funkcij $C^\infty([a,b])$, definiran z $f \in V \Leftrightarrow f^{(n)}(a) = f(b)^{(n)} = 0 \quad \forall n \in \mathbb{N}_0$. $D\colon C^\infty([a,b]) \to C^\infty([a,b])$ $Df = f'$

V tem primeru je $D^* = -D$.

6. Pokaži, da je $\ker(A^*A + B^*B) = \ker(A) \cap \ker(B)$ $(\ker(\sum_{j=1}^{n})A_j) = \bigcap_{j=1}^{n} \ker A_j)$

$$\ker(A^*A) = \ker(A) \quad \text{(predavanja)}$$

$\subset$

$\supseteq$ – trivialno

$\subseteq$

$$x \in \ker(A^*A) \Rightarrow x \in \ker(A)$$

$$0 = \langle A^*Ax, x \rangle = \langle Ax, Ax \rangle = \|Ax\|^2 \Rightarrow Ax = 0$$

$$\ker(A^*A + B^*B) = \ker(A) \cap \ker(B)$$

$\supseteq$ trivialno

$\subseteq$

$$(A^*A + B^*B)x = 0$$

$$0 = \langle (A^*A + B^*B)x, x \rangle = \langle A^*Ax, x \rangle + \langle B^*Bx, x \rangle =$$
$$\langle Ax, Ax \rangle + \langle Bx, Bx \rangle = \|Ax\|^2 + \|Bx\|^2 \Rightarrow Ax = 0 \text{ in } Bx = 0$$

Splošno

$$0 = \langle \sum_{j=1}^{n} A_j^* A_j x, x \rangle = \sum_{j=1}^{n} \langle A_j^* A_j x, x \rangle = \sum_{j=1}^{n} \langle A_j x, A_j x \rangle = \sum_{j=1}^{n} \|A_j x\|^2 = 0$$

$$\Rightarrow \|A_j x\| = 0 \ \forall j \Rightarrow A_j x = 0 \ \forall j.$$

7. Naj bo $A\colon V \to V$ in W invarianten za A. Pokaži, da je $W^{\perp}$ invarianten za A^*.

$\forall w \in W \quad Aw \in W$

Želimo pokazati $\quad \forall x \in W^{\perp} \quad A^* x \in W^{\perp}$

$x \in W^{\perp} \iff \langle x, w \rangle = 0 \quad \forall w \in W$

$x \in W^{\perp}, \quad \langle \underset{w \in W}{A^* x, w} \rangle = \langle \underset{x \in W^{\perp}}{x}, \underset{Aw \in W}{Aw} \rangle = 0 \Rightarrow A^* x \in W^{\perp}$

Bonus naloga

Naj bo $A \in M_n(\mathbb{R})$ in $A^T = -A$.
Pokaži, da ima A samo imaginarne lastne vrednosti.
Pokaži, da je $I - A$ obrnljiva matrika.

Najprej gledamo A kot element $M_n(\mathbb{C})$.

Opazimo $A^H = A^T$. Naj bo $v \in \mathbb{C}^n$ lastni vektor

standardni skalarni produkt

$\langle Av, v \rangle = \langle \lambda v, v \rangle \qquad \bar{\lambda} = -\lambda \Rightarrow \mathrm{Re}(\lambda) = 0.$

$\|$

$\langle v, A^H v \rangle = \langle v, -Av \rangle = -\langle v, Av \rangle = -\bar{\lambda} \langle v, v \rangle$

B je obrnljiva $\Leftrightarrow$ 0 ni lastna vrednost B.

Naj bo 0 lastna vrednost $I-A$.

$\exists x \neq 0 \quad (I-A)x = 0 \Rightarrow x = Ax$ $\quad$ 1 je lastna vrednost A

$Re(1) = 1 \neq 0 \rightarrow\leftarrow$