Kdaj je $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ normalna

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} \bar a & \bar b \\ \bar c & \bar d \end{bmatrix}^T = \begin{bmatrix} \bar a & \bar c \\ \bar b & \bar d \end{bmatrix}$$

$$AA^* = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \bar a & \bar c \\ \bar b & \bar d \end{bmatrix} = \begin{bmatrix} a\bar a + b\bar b & a\bar c + b\bar d \\ c\bar a + d\bar b & c\bar c + d\bar d \end{bmatrix}$$

$$A^*A = \begin{bmatrix} \bar a & \bar c \\ \bar b & \bar d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \bar a a + \bar c c & \bar a b + \bar c d \\ \bar b a + \bar d c & \bar b b + \bar d d \end{bmatrix}$$

Torej velja $AA^* = A^*A$ natanko tedaj ko je

$$\cancel{a\bar a} + b\bar b = \cancel{\bar a a} + \bar c c \qquad a\bar c + b\bar d = \bar a b + \bar c d$$

$$c\bar a + d\bar b = \bar b a + \bar d c \qquad c\bar c + \cancel{d\bar d} = \bar b b + \cancel{\bar d d}$$

$$\Leftrightarrow \quad b\bar b = c\bar c \quad \text{in} \quad \cancel{a}\bar c + b\bar d = \bar a b + \bar c \cancel{d}$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$|b| = |c| \qquad \bar c(a-d) = b(\bar a - \bar d)$$

Poseben primer: $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ je normalna

$\begin{bmatrix} 1 & i \\ i & 0 \end{bmatrix}$ ni normalna

Pozor: Iz $A = A^T$ ne sledi, da je A normalna

$\begin{bmatrix} 1 & i \\ i & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & i \\ i & 0 \end{bmatrix}$, toda $\begin{bmatrix} 1 & i \\ i & 0 \end{bmatrix}$ ni normalna

Dokazujmo, da za normalno matriko A

iz $Av = \lambda v$ sledi $A^* v = \bar{\lambda} v$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$(A-\lambda I)v=0 \qquad\qquad (A^*-\bar{\lambda} I)v=0$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$v \in \ker(A-\lambda I) \qquad\qquad v \in \ker(A^*-\bar{\lambda} I)$$

Zadošča dokazati, da velja

$$\ker(\underbrace{A-\lambda I}_{B}) = \ker(\underbrace{A^*-\bar{\lambda} I}_{B^*})$$

Dokazati torej moramo, da za normalno matriko B velja $\ker B = \ker B^*$

Upoštevamo $\ker B^*B = \ker B$ (zadnjič)

Odtod sledi (če vstaviš B^* namesto B), da velja

$\ker(BB^*) = \ker B^*$

Iz $BB^* = B^*B$ potem sledi

$$\underset{\ker B^*}{\underset{\parallel}{\ker BB^*}} = \underset{\ker B}{\underset{\parallel}{\ker B^*B}}$$

Torej je res $\ker B^* = \ker B$

A normalna

$\lambda_1, \ldots, \lambda_k$ vse paroma različne lastne vrednosti matrike A. Vemo, da je

$$① \ \mathbb{C}^n = \underbrace{\ker(A-\lambda_1 I)}_{B_1 \text{ ONB za}} \oplus \ldots \oplus \underbrace{\ker(A-\lambda_k I)}_{B_k \text{ je ONB za}}$$

(ker se da A diagonalizirati)

② Vemo tudi, da je $\ker(A-\lambda_i I) \perp \ker(A-\lambda_j I)$ če $i \neq j$

Naj bo B_i ortonormirana baza za $\ker(A-\lambda_i I)$

Iz ① in ② sledi, da je $\underbrace{B_1 \cup \ldots \cup B_k}_{\{v_1, \ldots, v_n\}}$ ONB za $\mathbb{C}^n$

Definiramo $P = [v_1, \ldots, v_n]$. Ta matrika je unitarna, ker velja $P^*P =$

$$= [v_1 \ldots v_n]^* [v_1 \ldots v_n]$$

$$= \begin{bmatrix} v_1^* \\ \vdots \\ v_n^* \end{bmatrix} [v_1 \ldots v_n] = \begin{bmatrix} v_1^* v_1 & \ldots & v_1^* v_n \\ \vdots & & \\ v_n^* v_1 & \ldots & v_n^* v_n \end{bmatrix}$$

$$= \begin{bmatrix} \langle v_1, v_1 \rangle & \ldots & \langle v_n, v_1 \rangle \\ \vdots & & \\ \langle v_1, v_n \rangle & \ldots & \langle v_n, v_n \rangle \end{bmatrix} \overset{\text{ONB}}{=} \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = I$$

Ponovno standardni skalarni produkt

$u = (\alpha_1, \dots, \alpha_n)$ $\qquad$ $v = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}$

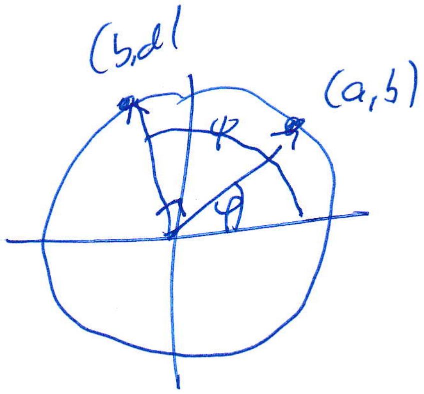
$v = (\beta_1, \dots, \beta_n)$

$\langle u, v \rangle = \alpha_1 \overline{\beta_1} + \dots + \alpha_n \overline{\beta_n}$ $\qquad$ $v^T \; [\beta_1 \dots \beta_n]$

$= \underbrace{[\overline{\beta_1} \dots \overline{\beta_n}]}_{v^*} \underbrace{\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}}_{u}$ $\qquad$ $v^* = \overline{v}^T = [\overline{\beta_1} \dots \overline{\beta_n}]$

$\Rightarrow \langle u, v \rangle = v^* u$

Ponovno definicijo unitarne matrike.
To je taka kvadratna matrika nad $\mathbb{C}$,
katere stolpci so ortonormirana baza
glede na standardni skalarni produkt v $\mathbb{C}^n$
(to je ekvivalentno z $A^* A = I$
Upoštevamo, da je to ekvivalentno z $A^*A = I$
(upoštevamo, da je to ekvivalentno: $A^* A = I \Leftrightarrow$ stolpci A so ONM)
ker je A kvadratna matrika, odtod sledi,
da je A obrnljiva in da velja $AA^* = I$
$(A^* A = I \Rightarrow A^{-1} = A^* \Rightarrow I = AA^{-1} = AA^*)$
Za unitarno matriko torej velja $AA^* = A^* A = I$,
torej je normalna.



Klasifikacija ortogonalnih

2x2 matrik

Kot med obema vektorjema je pravi

Kaj vemo o determinanti unitarne matrike

$$AA^* = I \qquad\qquad A^* = \overline{A}^T$$

$$\Rightarrow \underbrace{\det A \cdot \det A^*}_{\det A^* = \overline{\det A}} = \underbrace{\det I}_{= 1}$$

$$\det A^* = \det(\overline{A})^T = \det \overline{A} = \overline{\det A}$$

$$\Rightarrow \det A \cdot \overline{\det A} = 1$$

$$\Rightarrow |\det A| = 1 \quad \not\Rightarrow \det A = 1$$