

VEKTORSKA ANALIZA

$f: \mathbb{R}^3 \rightarrow \mathbb{R}$ SKALARNO POLJE (temperatura)

$$(x, y, z) \mapsto f(x, y, z)$$

$\vec{F}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ VEKTORSKO POLJE (polje sil)

$$(x, y, z) \mapsto (P(x, y, z), Q(x, y, z), R(x, y, z))$$

Operator $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$

Gradient: skalarno polje $\rightarrow$ vektorsko polje

$$\text{grad}(f) = \vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = (f_x, f_y, f_z)$$

Divergenca: vektorsko polje $\rightarrow$ skalarno polje

$$\text{div } \vec{F} = \vec{\nabla} \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

če je $\text{div}(\vec{F}) = 0$, je $\vec{F}$ brez izvorov

Rotor: vektorsko polje $\rightarrow$ vektorsko polje

$$\text{rot } \vec{F} = \vec{\nabla} \times \vec{F} = (R_y - Q_z, P_z - R_x, Q_x - P_y)$$

če je $\text{rot}(\vec{F}) = 0$, je $\vec{F}$ brez vrtilcev

Vektorsko polje $\vec{F}$ je POTENCIALNO, če obstaja skalarno polje f , da je $\text{grad } f = \vec{F}$. Takemu f če obstaja, pravimo POTENCIAL polja $\vec{F}$.

Izrek: $\text{rot}(\text{grad } f) = \vec{0}$

$$\text{div}(\text{rot } \vec{F}) = 0$$

Na dovolj lepih območjih velja tudi obrat.

Laplaceov operator Δ : skalarno polje $\xrightarrow{\Delta}$ skalarno polje

$$\Delta f = \text{div grad } f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Skalarno polje f je HARMONICNO, če velja $\Delta f = 0$.

U: $\vec{A} = (2yz, -x^2y, xz^2) = (P, Q, R)$ $P(x,y,z) = 2yz, Q = -x^2y, R = xz^2$
 $u = 2x^2yz^3$

Izračunaj $\text{div} \vec{A} \cdot u, \vec{A} \cdot \text{grad} u, \text{div}(u\vec{A}), \text{grad} u^2$.

$$\text{div} \vec{A} = \vec{\rho} \cdot \vec{A} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0 - x^2 + 2xz$$

$$\text{grad} u = (u_x, u_y, u_z) = (4xyz^3, 2x^2z^3, 6x^2yz^2)$$

$$\text{div} \vec{A} \cdot u = (-x^2 + 2xz) \cdot 2x^2yz^3 = -2x^4yz^3 + 4x^3yz^4$$

$$\begin{aligned} \vec{A} \cdot \text{grad} u &= (2yz, -x^2y, xz^2) \cdot (4xyz^3, 2x^2z^3, 6x^2yz^2) = \\ &= 8xy^2z^4 - 2x^4yz^3 + 6x^3yz^4 \end{aligned}$$

$$u \cdot \vec{A} = (4x^3y^2z^4, -2x^4y^2z^3, 2x^3yz^5)$$

$$\text{div}(u\vec{A}) = 8xy^2z^4 - 4x^4yz^3 + 10x^3yz^4 = \text{div} \vec{A} \cdot u + \vec{A} \cdot \text{grad} u$$

To velja tudi v splošnem:

$$\boxed{\text{div}(\vec{f} \cdot \vec{F}) = \text{div} \vec{F} \cdot \vec{f} + \vec{F} \cdot \text{grad} \vec{f}}$$

↑
skalarni produkt

$$u^2 = 4x^4y^2z^6$$

$$\text{grad} u^2 = (16x^3y^2z^6, 8x^4yz^6, 24x^4y^2z^5)$$

(N): Določite $a, b, c \in \mathbb{R}$, da bo polje $\vec{H} = (\underbrace{axy + z^3}_A, \underbrace{3x^2 + bz}_B, \underbrace{cxz^2 - y}_C)$ brez vrtincev. Izračunajte $\text{rot}(u\vec{H})$ in $\vec{H} \times \text{grad}u$, kjer je $u = yz^2$.

$\vec{H}$ brez vrtincev, če $\text{rot} \vec{H} = \vec{0}$

$$\text{rot} \vec{H} = (C_y - B_z, A_z - C_x, B_x - A_y) =$$

$$= (-1 - b, 3z^2 - cz^2, 6x - ax) = 0 \Rightarrow \begin{matrix} b = -1 \\ c = 3 \\ a = 6 \end{matrix}$$

$$u\vec{H} = (\underbrace{axy^2z^2 + yz^5}_A, \underbrace{3x^2yz^2 + byz^3}_B, \underbrace{cxy^2z^2 - y^2z^2}_C)$$

$$\text{rot}(u\vec{H}) = (C_y - B_z, A_z - C_x, B_x - A_y) =$$

$$= (cxy^4 - 2yz^2 - 6x^2yz - 3byz^2, 2xay^2z + 5yz^4 - cyz^4, 6xy^2z^2 - 2axy^2z^2 - z^5)$$

$$a=6, b=-1, c=3 \Rightarrow (3xy^4 - 2yz^2 - 6x^2yz + 3yz^2, 12xy^2z + 5yz^4 - 3yz^4, 6xy^2z^2 - 12xy^2z^2 - z^5) =$$

$$= (3xy^4 - 6x^2yz + yz^2, 12xy^2z + 2yz^4, -6xy^2z^2 - z^5)$$

$$\text{grad} u = (0, z^2, 2yz)$$

$$\vec{H} = (\underbrace{axy + z^3}_A, \underbrace{3x^2 + bz}_B, \underbrace{cxz^2 - y}_C)$$

$$\vec{H} \times \text{grad} u = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \\ 0 & z^2 & 2yz \end{vmatrix}$$

$$= (6x^2yz - 2yz^2 - 3xz^4 + yz^2, 0 - 12xy^2z - 2yz^4, 6xy^2z^2 + z^5 - 0)$$

Velja $\boxed{\text{rot}(u\vec{H}) = u \cdot \text{rot} \vec{H} - \vec{H} \times \text{grad} u}$ tudi v splošnem!

U: Izračunajte $\text{rot} \left(2xe^{\frac{y}{z}}, \frac{x^2}{z}e^{\frac{y}{z}}, -\frac{x^2y}{z^2}e^{\frac{y}{z}} \right)$

1. možnost: na roke (DU)

2. možnost: Opazimo:

$$2xe^{\frac{y}{z}} = \frac{\partial}{\partial x} (x^2e^{\frac{y}{z}})$$

$$\frac{x^2}{z}e^{\frac{y}{z}} = \frac{\partial}{\partial y} (x^2e^{\frac{y}{z}})$$

$$-\frac{x^2y}{z^2}e^{\frac{y}{z}} = \frac{\partial}{\partial z} (x^2e^{\frac{y}{z}})$$

$$\Rightarrow \text{rot}(\dots) = \text{rot}(\text{grad}(x^2e^{\frac{y}{z}})) = 0$$

U: Naj bo $r = \sqrt{x^2+y^2+z^2}$.

a) Izračunajte Δr .

b) Pri katerem eksponentu p je polje r^p harmonično?

$$\text{a) } \Delta r = \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} =$$

$$= \frac{3}{\sqrt{x^2+y^2+z^2}} - \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}} =$$

$$= \frac{3}{\sqrt{x^2+y^2+z^2}} - \frac{1}{\sqrt{x^2+y^2+z^2}} =$$

$$= \frac{2}{\sqrt{x^2+y^2+z^2}} =$$

$$= \frac{2}{r}$$

$$\frac{\partial r}{\partial x} = \frac{1}{\sqrt{x^2+y^2+z^2}} \cdot 2x$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{1}{\sqrt{x^2+y^2+z^2}} + -\frac{1}{\sqrt{x^2+y^2+z^2}} \cdot x \cdot \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} \cdot 2x$$

$$\text{Podobno } \frac{\partial r}{\partial y}, \frac{\partial^2 r}{\partial y^2}, \frac{\partial r}{\partial z} \text{ in } \frac{\partial^2 r}{\partial z^2}$$

$$\begin{aligned}
 b) \operatorname{grad}(r^p) &= \operatorname{grad}\left((x^2+y^2+z^2)^{\frac{p}{2}}\right) = \\
 &= \left(\frac{p}{2}(x^2+y^2+z^2)^{\frac{p}{2}-1} \cdot 2x, \frac{p}{2}(x^2+y^2+z^2)^{\frac{p}{2}-1} \cdot 2y, \frac{p}{2}(x^2+y^2+z^2)^{\frac{p}{2}-1} \cdot 2z\right) = \\
 &= p \cdot (x^2+y^2+z^2)^{\frac{p-2}{2}} \cdot (x, y, z) = \\
 &= p \cdot r^{p-2} \cdot (x, y, z)
 \end{aligned}$$

$$\begin{aligned}
 \Delta(r^p) &= \Delta\left((x^2+y^2+z^2)^{\frac{p}{2}}\right) = \operatorname{div}(\operatorname{grad}(r^p)) = \\
 &= p \cdot (x^2+y^2+z^2)^{\frac{p}{2}-1} \cdot \overset{\text{za } x, y \text{ in } z}{3} + p \cdot \left(\frac{p}{2}-1\right) \cdot (x^2+y^2+z^2)^{\frac{p}{2}-2} \cdot (2x^2+2y^2+2z^2) = \\
 &= p \cdot r^{p-2} \cdot 3 + p \cdot (p-2) \cdot r^{p-4} \cdot r^2 = \\
 &= p \cdot r^{p-4} \cdot (3r^2 + (p-2) \cdot r^2) = \\
 &= p \cdot r^{p-2} \cdot (3+p-2) = \\
 &= \underline{\underline{p \cdot (p+1) \cdot r^{p-2}}}
 \end{aligned}$$

$$\Delta(r^p) = 0 \Leftrightarrow \underline{\underline{p=0 \text{ ali } p=-1.}}$$

11: Poiščite vektorsko polje, čigar rotor je vektorsko polje $(x, -y, 0)$.

Namig: z komponento lahko postavimo na 0.

$\vec{F} = (P, Q, R)$ iskano vektorsko polje

$$\text{rot } \vec{F} = (x, -y, 0)$$

Namig: $\vec{F} = (P, Q, 0)$

$$\text{rot } \vec{F} = (-Q_z, P_z, Q_x - P_y) = (x, -y, 0)$$

$$Q_z = -x \rightarrow Q(x, y, z) = -xz + q(x, y)$$

$$P_z = -y \rightarrow P(x, y, z) = -yz + p(x, y)$$

$$Q_x = P_y \rightarrow \cancel{-z} + q_x(x, y) = \cancel{-z} + p_y(x, y)$$

$$q_x(x, y) = p_y(x, y) \rightarrow \text{najlepša možnost:}$$

$$p(x, y) = q(x, y) = 0$$

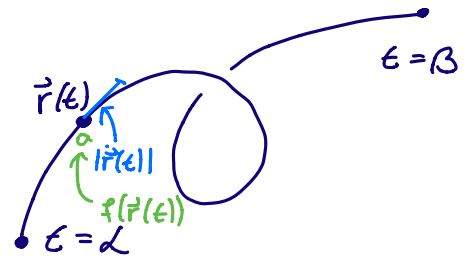
(v resnici imamo za p in q neskončno možnosti)

$$\underline{\underline{\vec{F} = (-yz, -xz, 0)}}$$

KRIVULJNI IN PLOSKOVNI INTEGRAL

- Krivuljni integral skalarne polja f vzdolž krivulje γ s parametrizacijo $\vec{r} = \vec{r}(t)$, $t \in [\alpha, \beta]$:

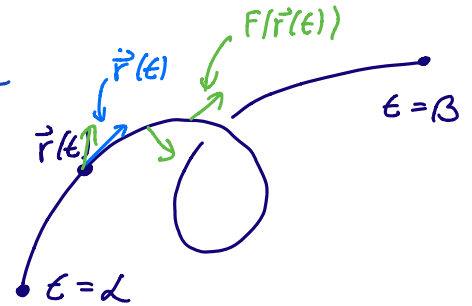
$$\int_{\gamma} f ds := \int_{\alpha}^{\beta} f(\vec{r}(t)) \cdot \|\vec{r}'(t)\| dt$$



* Definicija je neodvisna od izbrane parametrizacije $\vec{r} = \vec{r}(t)$.

- Krivuljni integral vektorskega polja $\vec{F}$ vzdolž krivulje γ s parametrizacijo $\vec{r} = \vec{r}(t)$, $t \in [\alpha, \beta]$:

$$\int_{\gamma} \vec{F} d\vec{s} := \int_{\alpha}^{\beta} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

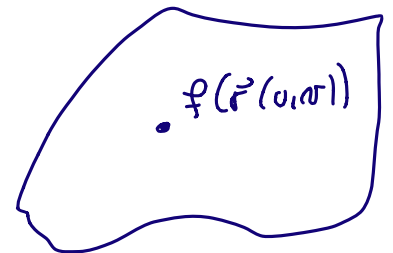


* Definicija je neodvisna od izbrane parametrizacije $\vec{r} = \vec{r}(t)$.

- Ploščovni integral skalarne polja f vzdolž ploskve P s parametrizacijo $\vec{r} = \vec{r}(u, v)$, $(u, v) \in D$:

$$\iint_P f dS = \iint_D f(\vec{r}(u, v)) \sqrt{EG - F^2} du dv$$

$\parallel \vec{r}_u \times \vec{r}_v \parallel$

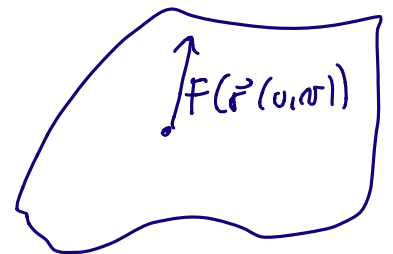


$$E = \vec{r}_u \cdot \vec{r}_u, F = \vec{r}_u \cdot \vec{r}_v, G = \vec{r}_v \cdot \vec{r}_v$$

- Ploščovni integral vektorskega polja $\vec{F}$ vzdolž ploskve P s parametrizacijo $\vec{r} = \vec{r}(u, v)$, $(u, v) \in D$:

$$\iint_P \vec{F} d\vec{S} = \iint_D \vec{F}(\vec{r}(u, v)) (\vec{r}_u \times \vec{r}_v) du dv$$

$\vec{r}_u \times \vec{r}_v$



1) Izračunajte $\int_C \frac{1}{x^2+y^2+z^2} ds$, če je C vijaznica s parametrizacijo

$$\vec{r}(t) = (a \cdot \cos t, a \cdot \sin t, b \cdot t) \quad \text{od } t=0 \text{ do } t=2\pi.$$

$$\dot{\vec{r}}(t) = (-a \cdot \sin t, a \cdot \cos t, b)$$

$$\|\dot{\vec{r}}(t)\| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + b^2} = \sqrt{a^2 + b^2}$$

$$\int_C \frac{ds}{x^2+y^2+z^2} = \int_0^{2\pi} \frac{1}{a^2 \cos^2 t + a^2 \sin^2 t + b^2 t^2} \cdot \sqrt{a^2 + b^2} dt =$$

$$= \sqrt{a^2 + b^2} \cdot \int_0^{2\pi} \frac{1}{a^2 + b^2 t^2} dt =$$

$$= \sqrt{a^2 + b^2} \cdot \int_0^{2\pi} \frac{\frac{1}{a^2}}{1 + \left(\frac{b}{a} t\right)^2} dt =$$

$$= \frac{\sqrt{a^2 + b^2}}{a^2} \cdot \int_0^{2\pi} \frac{1}{1 + \left(\frac{b}{a} t\right)^2} dt =$$

$$= \frac{\sqrt{a^2 + b^2}}{a^2} \cdot \frac{a}{b} \cdot \int_0^{\frac{b \cdot 2\pi}{a}} \frac{1}{1 + x^2} dx =$$

$$\begin{aligned} x &= \frac{b}{a} t \\ dx &= \frac{b}{a} dt \\ dt &= \frac{a}{b} dx \end{aligned}$$

$$= \frac{\sqrt{a^2 + b^2}}{a \cdot b} \cdot \arctan x \Big|_{x=0}^{\frac{2b\pi}{a}} =$$

$$= \underline{\underline{\frac{\sqrt{a^2 + b^2}}{a \cdot b} \cdot \arctan \frac{2b\pi}{a}}}$$

11: Izračunajte $\int_C y^2 ds$, če je C krivulja podana z enačbo $y = \sqrt{1-x^2}$, $z = 0$.

definirano za $x \in [-1, 1]$

$$\vec{r}(t) = (t, \sqrt{1-t^2}, 0)$$

$$\dot{\vec{r}}(t) = (1, \frac{1}{2} \cdot (1-t^2)^{-\frac{1}{2}} \cdot (-2t), 0) = (1, \frac{-t}{\sqrt{1-t^2}}, 0)$$

$$\|\dot{\vec{r}}(t)\| = \sqrt{1 + \frac{t^2}{1-t^2}} = \sqrt{\frac{1-t^2+t^2}{1-t^2}} = \frac{1}{\sqrt{1-t^2}}$$

$$\int_C y^2 ds = \int_{-1}^1 \sqrt{1-t^2} \cdot \frac{1}{\sqrt{1-t^2}} dt = \int_{-1}^1 \sqrt{1-t^2} dt = \int_{\pi}^0 \sqrt{1-\cos^2 x} \cdot (-\sin x) dx =$$

$t = \cos x$
 $dt = -\sin x dx$

$$= \int_0^{\pi} |\sin x| \cdot \sin x dx = \int_0^{\pi} \sin^2 x dx \stackrel{(*)}{=}$$

$\sin x \geq 0$ za $x \in [0, \pi]$

$$\stackrel{(*)}{=} 2 \cdot \int_0^{\frac{\pi}{2}} \sin^2 x dx =$$

$$= 2 \cdot \int_0^{\frac{\pi}{2}} \sin^{2 \cdot \frac{3}{2} - 1} x \cdot \cos^{2 \cdot \frac{1}{2} - 1} x dx =$$

$$= \beta\left(\frac{3}{2}, \frac{1}{2}\right) =$$

$$= \frac{\Gamma(\frac{3}{2}) \cdot \Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2} + \frac{1}{2})} = \frac{\frac{1}{2} \cdot \Gamma(\frac{1}{2}) \cdot \Gamma(\frac{1}{2})}{\Gamma(2)} = \frac{\pi}{2}$$

$$(*) : \int_0^{\frac{\pi}{2}} \sin^2 x dx \stackrel{\uparrow}{=} \int_{\pi}^{\frac{\pi}{2}} \sin^2(\pi - t) dt =$$

$$t = \pi - x \quad /'$$

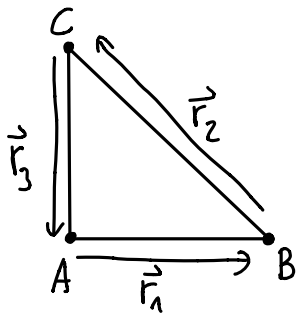
$$dt = -dx$$

$$= \int_{\frac{\pi}{2}}^{\pi} \sin^2 t dt =$$

$$= \int_{\frac{\pi}{2}}^{\pi} \sin^2 x dx$$

11: Izračunajte krivolinjski integral skalarne polja $u=x+y$ po robu trikotnika z oglišči $A(0,0)$, $B(1,0)$, $C(0,1)$.

$$\begin{aligned}\vec{r}_1(t) &= A + t \cdot (B-A) = (0+t \cdot (1-0), 0+t \cdot (0-0)) = (t, 0) & \dot{\vec{r}}_1 &= (1, 0) \\ \vec{r}_2(t) &= B + t \cdot (C-B) = (1+t \cdot (0-1), 0+t \cdot (1-0)) = (1-t, t) & \dot{\vec{r}}_2 &= (-1, 1) \\ \vec{r}_3(t) &= C + t \cdot (A-C) = (0+t \cdot (0-0), 1+t \cdot (0-1)) = (0, 1-t) & \dot{\vec{r}}_3 &= (0, -1)\end{aligned}$$



$$u(x,y) = x+y$$

$$\|\dot{\vec{r}}_1\| = 1$$

$$\|\dot{\vec{r}}_2\| = \sqrt{2}$$

$$\|\dot{\vec{r}}_3\| = 1$$

$$\begin{aligned}\int_C x+y \, ds &= \int_0^1 u(r_1(t)) \cdot \|\dot{\vec{r}}_1(t)\| \, dt + \int_0^1 u(r_2(t)) \cdot \|\dot{\vec{r}}_2(t)\| \, dt + \int_0^1 u(r_3(t)) \cdot \|\dot{\vec{r}}_3(t)\| \, dt = \\ &= \int_0^1 (t+0) \cdot 1 \, dt + \int_0^1 (1-\cancel{t}+\cancel{t}) \cdot \sqrt{2} \, dt + \int_0^1 (0+1-t) \cdot 1 \, dt = \\ &= \frac{t^2}{2} \Big|_0^1 + \sqrt{2} \cdot t \Big|_0^1 + \left(t - \frac{t^2}{2}\right) \Big|_0^1 = \\ &= \frac{1}{2} + \sqrt{2} + 1 - \frac{1}{2} = \\ &= \underline{\underline{1+\sqrt{2}}}\end{aligned}$$

U: Izračunajte integral vektorskega polja $\vec{R} = \begin{bmatrix} -\frac{1}{x} \\ \frac{y}{x} \end{bmatrix}$ po krivuljah

$K_1 = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$ in $K_2 = \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix}$, pri čemer gre parameter t obakrat od 0 do 1.

DN: Isto za $\vec{R} = \begin{bmatrix} 2xz \\ z^2 \\ 2yz + x^2 \end{bmatrix}$. R: $S_{K_1} = 2, S_{K_2} = 2$

$$\vec{r}_1 = (t, t, t) \quad \vec{r}_2 = (t, t^2, t^3)$$

$$\dot{\vec{r}}_1 = (1, 1, 1) \quad \dot{\vec{r}}_2 = (1, 2t, 3t^2)$$

$$\int_{K_1} \vec{R} d\vec{s} = \int_0^1 (t, -t, t) \cdot (1, 1, 1) dt = \int_0^1 (t - t + t) dt = \frac{t^2}{2} \Big|_0^1 = \underline{\underline{\frac{1}{2}}}$$

$$\begin{aligned} \int_{K_2} \vec{R} d\vec{s} &= \int_0^1 (t^2, -t^3, t) \cdot (1, 2t, 3t^2) dt = \int_0^1 (t^2 - 2t^4 + 3t^3) dt = \\ &= \left(\frac{t^3}{3} - 2\frac{t^5}{5} + 3\frac{t^4}{4} \right) \Big|_{t=0}^1 = \frac{1}{3} - \frac{2}{5} + \frac{3}{4} = \frac{20-24+45}{60} = \underline{\underline{\frac{41}{60}}} \end{aligned}$$

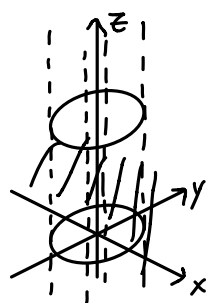
N: Izračunajte cirkulacijo vektorskega polja $\vec{R} = \begin{bmatrix} y \\ -z \\ x \end{bmatrix}$ po enem zavoju krivulje $\vec{r}(t) = (\cos t, \sin t, \cos t + \sin t)$

*CIRKULACIJA je integral po sklenjeni krivulji in jo označujemo z $\oint_C \vec{R} d\vec{r}$.

Je neodvisna od začetne oz. končne točke v parametrizaciji, odvisna pa je od orientacije.

$$\oint_C \vec{R} d\vec{r} = \int_0^{2\pi} (\sin t, -\cos t - \sin t, \cos t) \cdot (-\sin t, \cos t, -\sin t + \cos t) dt =$$

$$= \int_0^{2\pi} (-\sin^2 t - \cancel{\cos^2 t} - \sin t \cos t - \sin t \cos t + \cancel{\cos^2 t}) dt =$$



$$x^2 + y^2 = 1$$

$$z = x + y$$

$$= -\int_0^{2\pi} \sin^2 t dt - \int_0^{2\pi} 2 \sin t \cos t dt =$$

$$= -\int_0^{\pi} \sin^2 t dt - \int_{\pi}^{2\pi} \sin^2 t dt - \int_0^{2\pi} \sin 2t dt =$$

$$= -\int_0^{\pi} \sin^2 t dt + \int_{2\pi}^{\pi} \sin^2 t dt + \left. \frac{\cos 2t}{2} \right|_{t=0}^{2\pi} =$$

$$= -2 \int_0^{\pi} \sin^2 t dt + 0 =$$

od prejšnjih nalog:

$$\int_0^{\pi} \sin^2 t dt = \frac{\pi}{2} \rightarrow = -2 \cdot \frac{\pi}{2} = \underline{\underline{-\pi}}$$

N: Izračunajte krivuljni integral vektorskega polja $\vec{F} = (x^2, 2xy, yz)$ po krivulji $\vec{r} = (3t, t^2, t)$ od $T_0(0,0,0)$ do $T_1(3,1,1)$.

$$T_0 = \vec{r}(0)$$

$$T_1 = \vec{r}(1)$$

$$\dot{\vec{r}}(t) = (3, 2t, 1)$$

$$\begin{aligned}\int_C \vec{F} d\vec{r} &= \int_0^1 (9t^2, 2 \cdot 3t \cdot t^2, t^2 \cdot t) \cdot (3, 2t, 1) dt = \\ &= \int_0^1 27t^2 + 12t^4 + t^3 dt = \\ &= 27 \cdot \frac{1}{3} + 12 \cdot \frac{1}{5} + \frac{1}{4} = \\ &= 9 + \frac{12}{5} + \frac{1}{4} = \\ &= \frac{180 + 48 + 5}{20} = \underline{\underline{\frac{233}{20}}}\end{aligned}$$

N: Izračunajte $\int_C x dy + y dx$, če je C daljica od $A(a,0,0)$ do $B(0,b,0)$.

$$\text{Parametrizacija daljice: } \vec{r}(t) = A + t \cdot (B - A), \quad t \in [0,1]$$

$$\vec{r}(t) = (a, 0, 0) + t \cdot (0 - a, b, 0) = (a - ta, tb, 0)$$

$$\dot{\vec{r}}(t) = (-a, b, 0)$$

$$\begin{aligned}\int_C x dy + y dx &= \int_C (y, x, 0) \cdot (dx, dy, dz) = \int_0^1 (tb, a(1-t), 0) \cdot (-a, b, 0) dt = \\ &= \int_0^1 -abt + ab(1-t) dt = \left(abt - ab \frac{t^2}{2} \right) \Big|_{t=0}^1 = ab - ab = 0.\end{aligned}$$

Izrek: Krivuljni integral potencialnega polja $\vec{F} = \text{grad } u$ je odvisen le od začetne in končne točke poti γ .

$$\int_{\gamma} \vec{F} d\vec{s} = u(\vec{r}(B)) - u(\vec{r}(A))$$

N: Izračunajte $\int_{(0,0,0)}^{(1,1,1)} \vec{R} d\vec{r}$ za $\vec{R} = \begin{bmatrix} 2xz \\ z^2 \\ 2yz+x^2 \end{bmatrix}$.

Ali je $\vec{R}$ potencialno?

$$\text{rot } \vec{R} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz & z^2 & 2yz+x^2 \end{vmatrix} = (2z-2z, 2x-2x, 0-0) = (0,0,0)$$

$\vec{R}$ definiran na $\mathbb{R}^3$ (brez luknj) $\Rightarrow \vec{R}$ je potencialno

$$\vec{R} = \text{grad } u$$

$$u_x = 2xz$$

$$u = x^2z + a(y,z)$$

$$u = x^2z + z^2y + b(z)$$

$$u_y = z^2$$

$$\Rightarrow u_y = a_y = z^2$$

$$\Rightarrow u_z = x^2 + 2zy + b'(z) = 2yz + x^2$$

$$u_z = 2yz + x^2$$

$$a = z^2y + b(z)$$

$$b'(z) = 0 \Rightarrow b(z) = C \text{ konst.}$$

$$\Rightarrow u(x,y,z) = x^2z + z^2y + C$$

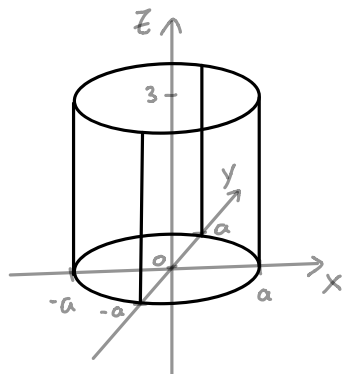
Potencial: $u = x^2z + z^2y$

$$\int_{(0,0,0)}^{(1,1,1)} \vec{R} d\vec{s} = u(1,1,1) - u(0,0,0) = 2 - 0 = \underline{\underline{2}}$$

* Primerjajte z rezultatom DN pri nalogi s tem $\vec{R}$ zgoraj!

PLOSKOVNI INTEGRAL SKALARNEGA POLJA

N: Izračunajte $\iint_P (x+z) dS$, če je P plašč valja $x^2+y^2=a^2$ med ravninama $z=0$ in $z=3$.



Parametrizacija:

$$\vec{r} = (a \cdot \cos \varphi, a \cdot \sin \varphi, z)$$

$$\vec{r}_\varphi = (-a \cdot \sin \varphi, a \cdot \cos \varphi, 0)$$

$$\vec{r}_z = (0, 0, 1)$$

$$E = \vec{r}_\varphi \cdot \vec{r}_\varphi = a^2 \sin^2 \varphi + a^2 \cos^2 \varphi = a^2$$

$$F = \vec{r}_\varphi \cdot \vec{r}_z = 0$$

$$G = \vec{r}_z \cdot \vec{r}_z = 1$$

$$\iint_P (x+z) dP = \int_0^{2\pi} \int_0^3 (a \cdot \cos \varphi + z) \cdot \sqrt{EG-F^2} dz d\varphi =$$

$$= \int_0^{2\pi} \int_0^3 (a \cdot \cos \varphi + z) \cdot \sqrt{a^2 - 0} dz d\varphi =$$

$$= \int_0^{2\pi} \int_0^3 (a \cdot \cos \varphi + z) \cdot a dz d\varphi =$$

$$= a \cdot \int_0^{2\pi} \left(a \cdot \cos \varphi \cdot z + \frac{z^2}{2} \right) \Big|_{z=0}^3 d\varphi =$$

$$= a \cdot \int_0^{2\pi} \left(3a \cdot \cos \varphi + \frac{9}{2} \right) d\varphi =$$

$$= 3a \cdot \left(a \cdot \sin \varphi \Big|_0^{2\pi} + \frac{3}{2} \varphi \Big|_0^{2\pi} \right) =$$

$$= 3a \cdot \left(0 + \frac{3}{2} \cdot 2\pi \right) = \underline{\underline{9a\pi}}$$

N: Izkazunaj površino ploskve $z = 2 - x^2 - y^2$, $z \geq 0$.

$$\vec{r}(x, y) = (x, y, 2 - x^2 - y^2) \quad ; \quad z = 2 - x^2 - y^2 \geq 0$$

$$\vec{r}_x(x, y) = (1, 0, -2x)$$

$$\vec{r}_y(x, y) = (0, 1, -2y)$$

$$E = \vec{r}_x \cdot \vec{r}_x = 1 + 4x^2$$

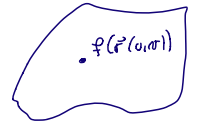
$$F = \vec{r}_x \cdot \vec{r}_y = 4xy$$

$$G = \vec{r}_y \cdot \vec{r}_y = 1 + 4y^2$$

• Ploskovni integral skalarnega polja φ vzdolž ploskve P s parametrizacijo $\vec{r} = \vec{r}(u, v)$, $(u, v) \in D$:

$$\iint_P \varphi dS = \iint_D \varphi(\vec{r}(u, v)) \|\vec{r}_u \times \vec{r}_v\| du dv$$

$$E = \vec{r}_u \cdot \vec{r}_u, F = \vec{r}_u \cdot \vec{r}_v, G = \vec{r}_v \cdot \vec{r}_v$$



$$EG - F^2 = 1 + 4x^2 + 4y^2 + 16x^2y^2 - 16x^2y^2 = 1 + 4(x^2 + y^2)$$

$$x = r \cdot \cos \varphi$$

$$y = r \cdot \sin \varphi$$

$$z = 2 - r^2$$

$$P = \iint_D \sqrt{EG - F^2} dx dy = \iint_D \sqrt{1 + 4(x^2 + y^2)} dx dy$$

$$= \int_0^{2\pi} d\varphi \int_0^{\sqrt{2}} \sqrt{1 + 4r^2} r dr =$$

$$\epsilon = 1 + 4r^2$$

$$d\epsilon = 8r dr$$

$$= 2\pi \cdot \int_1^9 \sqrt{\epsilon} \cdot \frac{1}{8} \frac{d\epsilon}{\epsilon} =$$

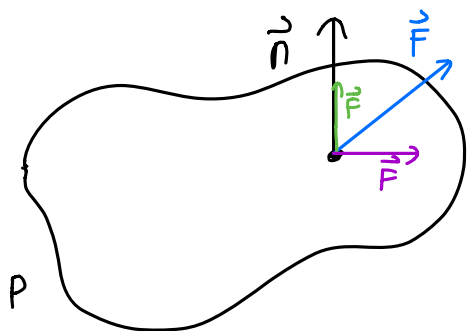
$$= \frac{2\pi}{8} \left[\frac{\epsilon^{\frac{3}{2}}}{\frac{3}{2}} \right]_{\epsilon=1}^9 =$$

$$= \frac{\pi}{4} \cdot \frac{2}{3} \cdot \left[\epsilon^{\frac{3}{2}} \right]_{\epsilon=1}^9 =$$

$$= \frac{\pi}{6} \cdot (27 - 1) =$$

$$= \frac{\pi}{6} \cdot (27 - 1) = \underline{\underline{\frac{13\pi}{3}}}$$

PLOSKOVNI INTEGRAL VEKTORSKEGA POLJA



$\vec{n} \parallel \vec{F}$ največji pretok

srednji pretok

$\vec{n} \perp \vec{F}$ najmanjši (ničelni) pretok

} skalarni produkt $\vec{n}$ in $\vec{F}$

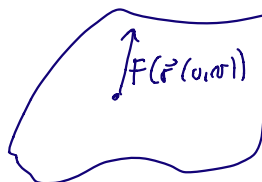
$$\int_P \vec{F} d\vec{S} = \pm \iint_D F(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) du dv$$

! smer pretoka !

: če je smer pretoka, smer $\vec{r}_u \times \vec{r}_v$, (+), sicer (-)

- Ploskovni integral vektorskega polja F vzdolž ploskve P s parametrizacijo $\vec{r} = \vec{r}(u,v)$, $(u,v) \in D$:

$$\iint_P \vec{F} d\vec{S} = \iint_D F(\vec{r}(u,v)) (\vec{r}_u \times \vec{r}_v) du dv$$



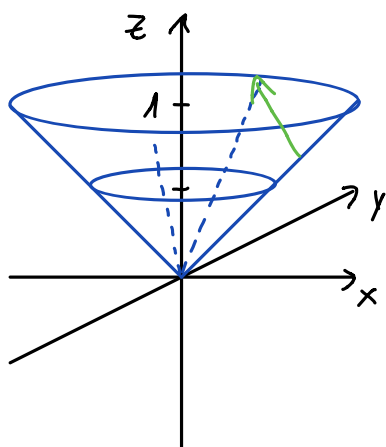
U: Izračunajte pretok polja $(x, y, 2z)$ skozi ploskev $\underbrace{z^2 = x^2 + y^2}_{\text{cilindrične koordinate}}; 0 \leq z \leq 1$ v smeri navzdol.

cilindrične koordinate

$$x = a \cdot \cos \varphi$$

$$y = a \cdot \sin \varphi$$

$$z^2 = x^2 + y^2 = a^2 \quad \begin{matrix} 0 \leq z \leq 1 \\ a \geq 0 \end{matrix} \rightarrow z = a$$



Parametrizacija ploskve:

$$\vec{r}(a, \varphi) = (a \cdot \cos \varphi, a \cdot \sin \varphi, a)$$

$$\varphi \in [0, 2\pi), a \in [0, 1]$$

$$\vec{r}_a(a, \varphi) = (\cos \varphi, \sin \varphi, 1)$$

$$\vec{r}_\varphi(a, \varphi) = (-a \cdot \sin \varphi, a \cdot \cos \varphi, 0)$$

$$\vec{r}_a \times \vec{r}_\varphi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos \varphi & \sin \varphi & 1 \\ -a \sin \varphi & a \cos \varphi & 0 \end{vmatrix} = \underline{\underline{(-a \cdot \cos \varphi, -a \cdot \sin \varphi, a)}}$$

(kaže navzgor) $\uparrow$

$$\Phi = \underbrace{-}_{\text{pretok}} \int_0^{2\pi} d\varphi \int_0^1 (a \cos \varphi, a \sin \varphi, 2a) \cdot (-a \cos \varphi, -a \sin \varphi, a) da =$$

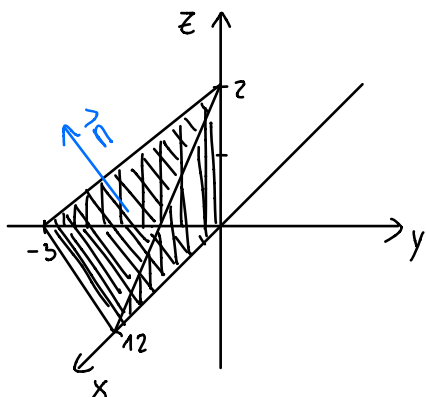
$$= - \int_0^{2\pi} d\varphi \int_0^1 (-a^2 \cos^2 \varphi - a^2 \sin^2 \varphi + 2a^2) da =$$

$$= - \int_0^{2\pi} d\varphi \int_0^1 a^2 da = -2\pi \left. \frac{a^3}{3} \right|_{a=0}^1 = \underline{\underline{-2\pi \cdot \frac{1}{3}}}$$

1) Izračunajte pretok vektorskega polja $\vec{F} = (x^2, y^2, z^2)$ skozi ploskev:

$$x - 2y + 3z = 6, \quad x > 0, y < 0, z > 0,$$

orientirano tako, da normala kaže navzgor



$$y=0: \quad 3z = 6 - x \\ z = 2 - \frac{x}{3}$$

$$x=0: \quad 3z = 6 + 2y \\ z = 2 + \frac{2}{3}y$$

$$\text{Eksplicitna oblika: } x = 6 + 2y - 3z$$

Območje D: $-3 \leq y < 0$ očitno iz skice

$$(x=0): \quad z = 2 + \frac{2}{3}y \Rightarrow 0 < z \leq 2 + \frac{2}{3}y$$

$$\text{Parametrizacija: } \vec{r}(y, z) = (6 + 2y - 3z, y, z)$$

$$\vec{r}_y(y, z) = (2, 1, 0)$$

$$\vec{r}_z(y, z) = (-3, 0, 1)$$

$$\vec{r}_y \times \vec{r}_z = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 0 \\ -3 & 0 & 1 \end{vmatrix} = (1, -2, 3)$$

↑
kaže navzgor

Pretok:

$$\Phi = + \int_{-3}^0 dy \int_0^{2+\frac{2}{3}y} ((6+2y-3z)^2, y^2, z^2) \cdot (1, -2, 3) dz = \dots = 15$$

$$= \int_{-3}^0 dy \int_0^{2+\frac{2}{3}y} (6+2y-3z)^2 - 2y^2 + 3z^2 dz =$$

$$= \int_{-3}^0 \left[\frac{(6+2y-3z)^3}{-3} - 2y^2 z + \frac{3z^3}{3} \right]_{z=0}^{2+\frac{2}{3}y} dy = \int_{-3}^0 \left[\frac{1}{9} (6+2y)^3 - 2y^2 (2+\frac{2}{3}y) + (2+\frac{2}{3}y)^3 \right] dy =$$

$$= \int_{-3}^0 3(2+\frac{2}{3}y)^3 - 2y^2(2+\frac{2}{3}y) + (2+\frac{2}{3}y)^3 dy = \int_{-3}^0 4(2+\frac{2}{3}y)^3 - 2y^2(2+\frac{2}{3}y) dy \quad \leftarrow t = 2 + \frac{2}{3}y \Rightarrow y = (t-2) \cdot \frac{3}{2} \right. \\ \left. dy = \frac{3}{2} dt \right.$$

$$= \int_0^2 (4t^3 - 2 \cdot (t-2)^2 \cdot (\frac{3}{2})^2 \cdot t) \cdot \frac{3}{2} dt = \frac{3}{2} \int_0^2 4t^3 - \frac{9}{2} (t^2 - 4t + 4) \cdot t dt = \frac{3}{2} \int_0^2 -\frac{1}{2} t^3 + 18t^2 - 18t dt =$$

$$= \frac{3}{2} \left(-\frac{t^4}{8} + 18 \frac{t^3}{3} - 18 \frac{t^2}{2} \right) \Big|_0^2 = \frac{3}{2} (-2 + 6 \cdot 8 - 9 \cdot 4) = 3 \cdot (-1 + 24 - 18) = 3 \cdot 5 = \underline{\underline{15}}$$

INTEGRALSKI IZREKI

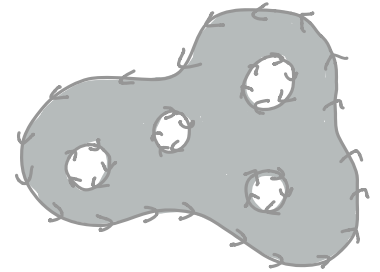
- Greenova formula
- Stokesov izrek
- Gaussov oz. divergenčni izrek

GREENOVA FORMULA:

Za regularno (rob = končno št. sklenjenih krivulj) območje $D \subseteq \mathbb{R}^2$ in pozitivno orientiran rob ∂D ter zvezno odvedljivo polje $\vec{F} = (P, Q) : D \rightarrow \mathbb{R}^2$ velja:

$$\int_{\partial D} \vec{F} d\vec{s} = \iint_D (Q_x - P_y) dx dy$$

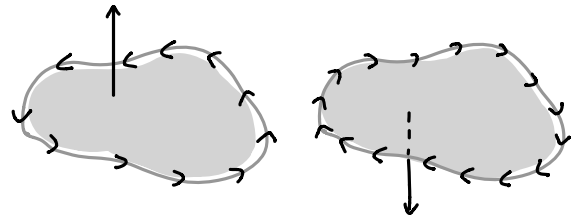
krivuljni integral $\longleftrightarrow$ dvojni integral



STOKESOV IZREK:

Naj bo $P \subseteq \mathbb{R}^3$ regularna ploskev s skladno orientiranim robom ∂P in naj bo $\vec{F}$ zvezno odvedljivo polje na okolici P .

$$\int_{\partial P} \vec{F} d\vec{s} = \iint_P \text{rot } \vec{F} d\vec{S}$$



GAUSSOV IZREK:

Naj bo $\Omega \subseteq \mathbb{R}^3$ omejeno, pozitivno orientirano in regularno območje ter $\vec{F}$ zvezno odvedljivo polje na okolici.

$$\iint_{\partial \Omega} \vec{F} d\vec{S} = \iiint_{\Omega} \text{div } \vec{F} dx dy dz$$

(11) Naj bo K rob trikotnika z oglišči $A(0,0)$, $B(1,0)$ in $C(0,1)$, orientiran v nasprotni smeri urinega kazalca. Izračunajte:

$$\oint_K (x dx + y dy),$$

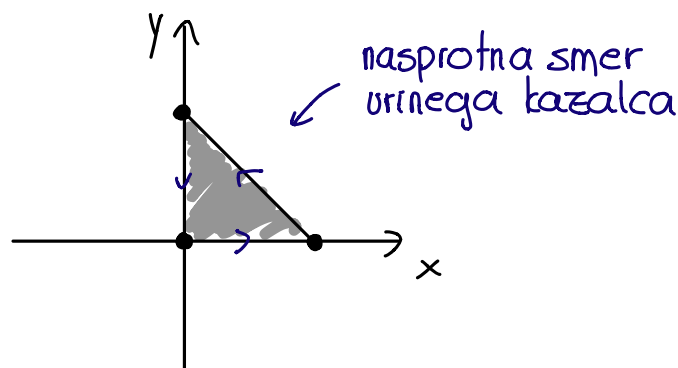
kjer je

$$x = \begin{cases} x \arctan \frac{x}{y} & ; y \neq 0 \\ \frac{\pi x}{2} & ; y = 0 \end{cases}$$

$$y = \begin{cases} y \arctan \frac{x}{y} & ; y \neq 0 \\ 0 & ; y = 0 \end{cases}$$

DU: Izračunajte po definiciji! (Krivuljo razdelite na tri dele)

Trikotnik:



$$P = x \quad P_y = x \cdot \frac{1}{1 + \frac{x^2}{y^2}} \cdot \left(-\frac{x}{y^2}\right) = -\frac{x^2}{y^2 + x^2}$$

$$Q = y \quad Q_x = y \cdot \frac{1}{1 + \frac{x^2}{y^2}} \cdot \left(\frac{1}{y}\right) = \frac{y^2}{y^2 + x^2}$$

$$Q_x - P_y = \frac{y^2}{y^2 + x^2} - \frac{-x^2}{y^2 + x^2} = 1$$

$$\iint_D 1 ds = pl(D)$$

$$\oint_K (x dx + y dy) = \oint_K (x, y) d\vec{s} \stackrel{\text{Greenova formula}}{=} \iint_{\triangle} 1 dx dy \stackrel{\downarrow}{=} pl(\triangle) = \frac{1}{2} \cdot 1 \cdot 1 = \underline{\underline{\frac{1}{2}}}$$

Greenova
formula

Po definiciji:

$$\oint_K (x, y) d\vec{s} = \int_{AB} (x, y) d\vec{s} + \int_{BC} (x, y) d\vec{s} + \int_{CA} (x, y) d\vec{s} = \frac{\pi}{4} + \left(\frac{1}{2} - \frac{\pi}{4}\right) + 0 = \underline{\underline{\frac{1}{2}}}$$

$$AB: \vec{r}_1(t) = (1-t) \cdot \vec{r}_A + t \cdot \vec{r}_B = (t, 0); \quad t \in [0, 1]$$

$$\int_{AB} (x, y) d\vec{s} = \int_0^1 \left(\frac{\pi t}{2}, 0\right) \cdot (1, 0) dt = \int_0^1 \frac{\pi t}{2} dt = \frac{\pi}{4}$$

$$BC: \vec{r}_2(t) = (1-t) \vec{r}_B + t \cdot \vec{r}_C = (1-t, t); \quad t \in [0, 1]$$

$$\begin{aligned} \int_{BC} (x, y) d\vec{s} &= \int_0^1 \left((1-t) \arctan \frac{1-t}{t}, t \arctan \frac{1-t}{t}\right) \cdot (-1, 1) dt = \\ &= \int_0^1 (2t-1) \arctan \frac{1-t}{t} dt \stackrel{\text{p.p.}}{=} \dots = \frac{1}{2} - \frac{\pi}{4} \end{aligned}$$

$v = \arctan \frac{1-t}{t}$
 $dv = (2t-1) dt$

$$CA: \vec{r}_3(t) = (1-t) \vec{r}_C + t \cdot \vec{r}_A = (0, 1-t); \quad t \in [0, 1]$$

$$\int_{CA} (x, y) d\vec{s} = \int_0^1 (0, 0) \cdot (0, -1) dt = 0$$

Dodatne naloge:

1. Izračunaj integral vektorskega polja $\vec{F}(x, y) = (2x^2 + y^2, (x + y)^2)$ po robu trikotnika z oglišči $(1, 1)$, $(2, 2)$ in $(1, 3)$ v pozitivni smeri.

Rešitev: $\frac{8}{3}$

2. Izračunaj integral vektorskega polja $\vec{F}(x, y) = (-x^2y, xy^2)$ po pozitivno orientirani krivulji

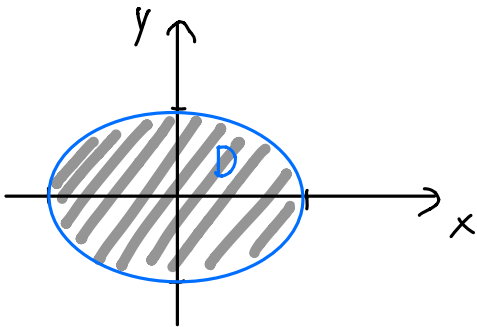
$$\{(x, y) : x^2 + y^2 = R^2, x \geq 0, y \geq 0\} \cup \{(0, y) : 0 \leq y \leq R\} \cup \{(x, 0) : 0 \leq x \leq R\}.$$

Rešitev: $\frac{\pi R^4}{8}$

3. S pomočjo Greenove formule izračunaj ploščino območja, ki ga omejuje elipsa z enačbo $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, kjer $a, b > 0$.

Rešitev: $ab\pi$

3.



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$pl(D) = ?$$

IDEJA:

$$pl(D) = \iint_D 1 \, dx \, dy = \xrightarrow{\text{Greenova formula}} \int_{\partial D} \vec{F} \, d\vec{s} \quad ; \quad \vec{F} = (P, Q)$$

(dvogni integral) (krivuljni integral)

kčemo $\vec{F}$, za katerega je $Q_x - P_y = 1$ (npr. primer, kjer je

$Q_x = 1 \text{ in } P_y = 0$

$\hookrightarrow Q = x \text{ in } P = 0$

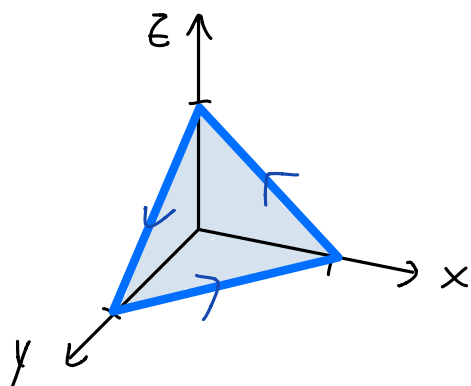
Parametrizacija elipse: $x = a \cdot \cos \varphi$
 $y = b \cdot \sin \varphi$
 $0 \leq \varphi < 2\pi$

$$pl = \iint_D 1 \, dx \, dy = \int_{\partial D} (0, x) \, d\vec{z} = \int_0^{2\pi} (0, a \cdot \cos \varphi) \cdot (-a \cdot \sin \varphi, b \cdot \cos \varphi) \, d\varphi = a \cdot b \cdot \pi$$

⑫: Izračunajte cirkulacijo :

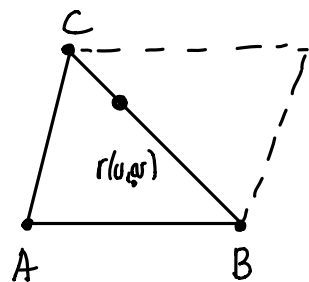
$$\oint_K \left(\left(\frac{1}{(1+x^2)^2} + y^2 - z^2 \right) dx + \left(\frac{1}{(1+y^2)^2} - x^2 + z^2 \right) dy + \left(\frac{1}{(1+z^2)^2} - x^2 - y^2 \right) dz \right),$$

kjer je K krivulja, ki gre od točke $(1,0,0)$ premočno do $(0,1,0)$, nato premočno do $(0,0,1)$ in nato premočno spet do $(1,0,0)$



* cirkulacija: krivolinijski integral po sklenjeni ploskvi

Trikotnik med tremi oglišči lahko parametriziramo :



$$\vec{r}(u,v) = \vec{r}_A + u \cdot \vec{AC} + v \cdot \vec{AB} ; \quad u, v \geq 0, \quad u+v \leq 1$$

⇒ na našem trikotniku :

$$\vec{r}(u,v) = (1,0,0) + u \cdot (-1,0,1) + v \cdot (-1,1,0) = (1-u-v, v, u)$$

$$\vec{r}_u(u,v) = (-1,0,1)$$

$$\vec{r}_v(u,v) = (-1,1,0)$$

$$\vec{r}_u \times \vec{r}_v(u,v) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{vmatrix} = (-1, -1, -1)$$

$$\oint_K \dots = \oint_K (x, y, z) d\vec{s} \quad ;$$

$$X = \frac{1}{(1+x^2)^2} + y^2 - z^2 \quad \frac{\partial X}{\partial y} = 2y \quad \frac{\partial X}{\partial z} = -2z$$

$$Y = \frac{1}{(1+y^2)^2} - x^2 + z^2 \quad \frac{\partial Y}{\partial x} = -2x \quad \frac{\partial Y}{\partial z} = 2z$$

$$Z = \frac{1}{(1+z^2)^2} - x^2 + y^2 \quad \frac{\partial Z}{\partial x} = -2x \quad \frac{\partial Z}{\partial y} = 2y$$

$$\oint_K (x, y, z) d\vec{s} \stackrel{\uparrow}{=} \pm \iint_{\Delta} \text{rot}(x, y, z) d\vec{S} =$$

Stokesov
izrek

$$= \pm \iint_{\Delta} \left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}, \frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}, \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right) d\vec{S} =$$

$$= \pm \iint_{\Delta} (2y - 2z, -2z + 2x, -2x - 2y) d\vec{S} =$$

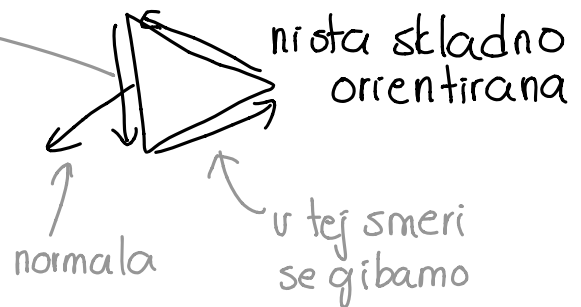
$$= \pm \iint_{\substack{0 \leq u \leq 1 \\ 0 \leq v \leq 1-u}} (2v - 2(1-u-v), -2(1-u-v) + 2u, -2u - 2v) \cdot (-1, -1, -1) du dv =$$

$$= \pm \iint_{\substack{0 \leq u \leq 1 \\ 0 \leq v \leq 1-u}} 2v - 2\cancel{1} + 2u - 2\cancel{(1-u-v)} + 2\cancel{(1-u-v)} + 2\cancel{v} du dv =$$

$$= \pm \iint_{\substack{0 \leq u \leq 1 \\ 0 \leq v \leq 1-u}} 4 \cdot v du dv =$$

$$= -4 \cdot \int_0^1 du \int_0^{1-u} v dv =$$

$$= -4 \cdot \int_0^1 u \cdot (1-u) du = -4 \cdot \left(\frac{u^2}{2} - \frac{u^3}{3} \right) \Big|_0^1 = -4 \cdot \left(\frac{1}{2} - \frac{1}{3} \right) = -4 \cdot \frac{1}{6} = \underline{\underline{-\frac{2}{3}}}$$



Dodatne naloge:

1. Izračunaj integral vektorskega polja $\vec{F}(x, y, z) = (x + 3y + 2z, 2x + z, x - y)$ po robu trikotnika z oglišči $A(2, 0, 0)$, $B(0, 3, 0)$ in $C(0, 0, 1)$, ki ga orientiramo v smeri $A \rightarrow B \rightarrow C \rightarrow A$.

Rešitev: -5

2. S pomočjo Stokesovega izreka izračunaj integral vektorskega polja

$$\vec{F}(x, y, z) = (x^2 y^3, 1, z)$$

po pozitivno orientirani krožnici z enačbo $x^2 + y^2 = R^2$, $z = 0$, kjer je $R > 0$.

Rešitev: $-\frac{\pi R^6}{8}$

- 3. S pomočjo Stokesovega izreka izračunaj integral vektorskega polja

$$\vec{F}(x, y, z) = (x + y, x + z, y + z)$$

po pozitivno orientirani krožnici, ki je presek sfere z enačbo $x^2 + y^2 + z^2 = a^2$ in ravnine z enačbo $x + y + z = 0$, kjer je $a > 0$.

Rešitev: 0

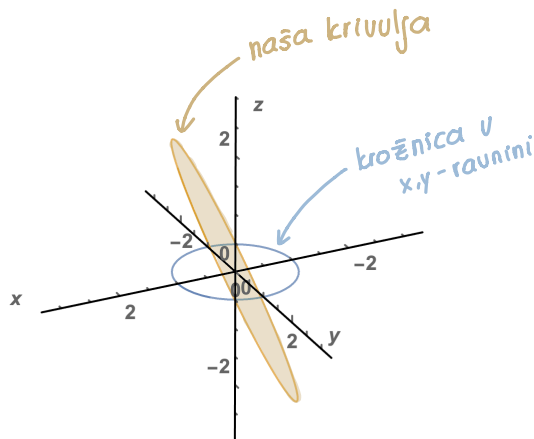
Naloga iz starega kolokvija

1. [30] Dana je prostorska krivulja: $\vec{r} = \begin{bmatrix} \cos t \\ \sin t \\ 2 \cos t - \sin t \end{bmatrix}$, ko gre t od 0 do 2π .

a) Skicirajte jo! Za katero znano krivuljo gre?

b) Izračunajte integral vektorskega polja $\vec{R} = \begin{bmatrix} y \sin^2 z \\ -x \cos^2 z \\ 2xy \sin z \cos z \end{bmatrix}$ po dani krivulji.

Namig: Stokesov izrek.



Ploskev, ki jo omejuje naša krivulja

$$z = 2x - y \quad \text{in} \quad x^2 + y^2 \leq 1$$

preseka ravnine z valjem

Parametrizacija ploskve

$$x = r \cos \varphi$$

$$y = r \sin \varphi$$

$$z = r(2 \cos \varphi - \sin \varphi)$$

$$; \varphi \in [0, 2\pi), r \in [0, 1]$$

1) S pomočjo Gaussovega izreka izračunajte pretok vektorskega polja:

$$x = x(y^2 + z^2) \quad y = y(x^2 + z^2) \quad z = z(x^2 + y^2)$$

skozi rob enotske krogle $x^2 + y^2 + z^2 = 1$, orientiran navzven (iztok iz krogle).

pogoji izpolnjeni: K omejeno območje ✓
 K regularno območje ✓
 (x, y, z) ev. odv. polje ✓

$$\Phi = \iint_{\partial K} (x, y, z) \cdot d\vec{S} \stackrel{\text{Gauss}}{=} \iiint_K \operatorname{div} (x, y, z) \, dV =$$

$$= \iiint_K (y^2 + z^2) + x^2 + z^2 + x^2 + y^2 \, dx \, dy \, dz =$$

$$= 2 \iiint_K x^2 + y^2 + z^2 \, dx \, dy \, dz =$$

Spomnimo se uvedbe novih spremenljivk v trojnem integralu

$$= 2 \cdot \int_0^1 dr \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2\pi} r^2 \cdot |r^2 \cos \theta| \, d\varphi =$$

upoštevamo, da je $\cos \theta \geq 0$ za $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$= 2 \cdot \int_0^1 dr \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^4 \cos \theta \cdot 2\pi \, d\theta =$$

$$= 4\pi \cdot \int_0^1 r^4 \cdot \sin \theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dr =$$

$$= 4\pi \cdot 2 \cdot \int_0^1 r^4 \, dr =$$

$$= 8\pi \cdot \frac{r^5}{5} \Big|_0^1 = \underline{\underline{\frac{8\pi}{5}}}$$

K (sferične koordinate):

$$x = r \cdot \cos \varphi \cdot \cos \theta$$

$$y = r \cdot \sin \varphi \cdot \cos \theta$$

$$z = r \cdot \sin \theta$$

$$r \in [0, 1]$$

$$\varphi \in [0, 2\pi]$$

$$\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

$$J = r^2 \cdot \cos \theta$$

2) Izračunajte iztok vektorskega polja $\underbrace{\begin{bmatrix} x + \sqrt{y^2 + z^2} \\ y + \sqrt{x^2 + z^2} \\ z + \sqrt{x^2 + y^2} \end{bmatrix}}_{\vec{F}}$ iz stožca $z^2 \geq x^2 + y^2$, $0 \leq z \leq 1$.

! DN R: $\Phi = \pi$

$$\int_{\partial T} \vec{F} \cdot d\vec{S} = \iiint_T \operatorname{div} \vec{F} \, dV = \iiint_T 3 \, dV = 3 \cdot V = 3 \cdot \frac{\pi}{3} = \pi$$

$$\operatorname{div} \vec{F} = 1 + 1 + 1 = 3$$

Cilindrične koordinate:

$$V = \int_0^{2\pi} d\varphi \int_0^1 dz \int_0^z r \, dr = 2\pi \cdot \int_0^1 \frac{r^2}{2} \Big|_0^z dz = \pi \cdot \frac{z^3}{3} \Big|_0^1 = \frac{\pi}{3}$$

Dodatne naloge:

1. Izračunaj pretok vektorskega polja $\vec{F}(x, y, z) = (x, y, z)$ navzven skozi površino telesa, ki ima volumen V .

Rešitev: $3V$

2. S pomočjo Gaussovega izreka izračunaj pretok vektorskega polja

$$\vec{F}(x, y, z) = (x^3, y^3, z^3)$$

navzven iz sfere z radijem $a > 0$.

Rešitev: $\frac{12\pi a^5}{5}$

3. S pomočjo Gaussovega izreka izračunaj pretok vektorskega polja

$$\vec{F}(x, y, z) = (z^2 - y^2, x^2 - z^2, y^2 - x^2)$$

navzven iz zgornje polovice sfere z radijem $a > 0$.

Rešitev: 0

4. S pomočjo Gaussovega izreka izračunaj pretok vektorskega polja

$$\vec{F}(x, y, z) = (x^3, y^3, z^3)$$

navzven iz sfere z radijem $a > 0$.

Rešitev: $\frac{12\pi a^5}{5}$

5. S pomočjo Gaussovega izreka izračunaj pretok vektorskega polja

$$\vec{F}(x, y, z) = (x^3, y^3, z^3)$$

navzven iz plašča stožca z enačbo $x^2 + y^2 \leq \frac{R^2}{h^2} z^2$, $0 \leq z \leq h$.

Rešitev: $\frac{\pi R^2 h (3R^2 - 4h^2)}{10}$

3. Namig:

ploskev ni sklenjena: dodamo ji krog v x, y -ravnini (pozrite na orientacije!)

$$\text{GAUSSOV IZREK: } \oint_{\partial V} \vec{F} d\vec{S} = \iiint_V \text{div } \vec{F} dV = \dots = 0$$

$\parallel$
 $\oint_0 \vec{F} d\vec{S} + \oint_{\text{O}} \vec{F} d\vec{S}$

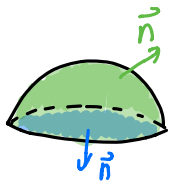
$$\Rightarrow \oint_{\text{O}} \vec{F} d\vec{S} = - \oint_0 \vec{F} d\vec{S}$$

tega znamo izračunati!

11: S pomočjo Gaussovega izreka izračunajte pretok vektorskega polja

$$\vec{F}(x, y, z) = (z^2 - y^2, x^2 - z^2, y^2 - x^2)$$

navzven iz zgornje polovice sfere z radijem $a > 0$.



T... zgornja polovica krogle

$$\partial T = P \cup K$$

↑ ↖
zgornja polovica krog
sfere

$$\int_{\partial T} \vec{F} d\vec{S} = \iiint_T \operatorname{div} \vec{F} dV$$

$$\int_P \vec{F} d\vec{S} + \int_K \vec{F} d\vec{S} = \iiint_T \operatorname{div} \vec{F} dV$$

$$\int_P \vec{F} d\vec{S} = \iiint_T \operatorname{div} \vec{F} dV - \int_K \vec{F} d\vec{S}$$

$$\operatorname{div} \vec{F} = 0 + 0 + 0 = 0$$

$$\int_P \vec{F} d\vec{S} = - \int_K \vec{F} d\vec{S}$$

Parametrizacija kroga: $x = r \cos \varphi$
 $y = r \sin \varphi$
 $z = 0$

$$\begin{aligned} \vec{r}(r, \varphi) &= (r \cos \varphi, r \sin \varphi, 0) \\ \vec{r}_r(r, \varphi) &= (\cos \varphi, \sin \varphi, 0) \\ \vec{r}_\varphi(r, \varphi) &= (-r \sin \varphi, r \cos \varphi, 0) \end{aligned}$$

$$\begin{aligned} r &\in [0, a] \\ \varphi &\in [0, 2\pi) \end{aligned}$$

$$\vec{r}_r \times \vec{r}_\varphi = (0 - 0, 0 - 0, r \cos^2 \varphi - (-r \sin^2 \varphi)) = (0, 0, r)$$

↑ kaže navzgor
(želimo navzdol)

$$= - \left(- \int_0^{2\pi} d\varphi \int_0^a (0^2 - r^2 \sin^2 \varphi, r^2 \cos^2 \varphi - 0^2, r^2 \sin^2 \varphi - r^2 \cos^2 \varphi) \cdot (0, 0, r) dr \right) =$$

$$= \int_0^{2\pi} d\varphi \int_0^a r^3 (\sin^2 \varphi - \cos^2 \varphi) dr$$

$$= \int_0^{2\pi} -\cos 2\varphi \cdot \frac{r^4}{4} \Big|_{r=0}^a d\varphi = -\frac{a^4}{4} \cdot (-\sin 2\varphi) \cdot \frac{1}{2} \Big|_0^{2\pi} = 0$$