DS 2 VAJE 11

1. $(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, +)$

EL. PARI, SEŠTEVAMO PO KOMPONENTAH

$$\mathbb{Z}_2 \times \mathbb{Z}_3 \overset{?}{\cong} \mathbb{Z}_6$$

IZOMORFIZEM : BIJEKTIVNI HOMOMORFIZEM

$$f(x+y) = f(x) + f(y)$$

$$|\mathbb{Z}_2 \times \mathbb{Z}_3| = 2 \cdot 3 = 6$$
$$|\mathbb{Z}_6| = 6$$

$$f : \mathbb{Z}_6 \to \mathbb{Z}_2 \times \mathbb{Z}_3$$

$$f(0) = (0,0)$$
$$f(1) = (0,1)$$

NI OK

$$f(2) = f(1+1) = f(1) + f(1) = (0,1) + (0,1) = (0,2)$$
$$f(3) = f(2+1) = f(2) + f(1) = (0,2) + (0,1) = (0,0)$$

NI BIJEKCIJA

$$f(1) = (1, 1)$$
$$f(2) = 2 \cdot (1,1) = (0,2)$$
$$f(3) = 3 \cdot (1,1) = (1,0)$$
$$f(4) = 4 \cdot (1,1) = (0,1)$$
$$f(5) = 5(1,1) = (1,2)$$

$\Rightarrow$ JE BIJEKCIJA

$$f(x) = x \cdot (1,1)$$

$$f(x+y) = (x+y) \cdot (1,1) = ((x+y) \bmod 2, (x+y) \bmod 3)$$

$$f(x) + f(y) = x \cdot (1,1) + y \cdot (1,1) = ((x+y) \bmod 2, (x+y) \bmod 3)$$

$\Rightarrow$ JE HOMOMORFIZEM

2. $\mathbb{Z}_2 \times \mathbb{Z}_4 \cong \mathbb{Z}_8$

$$f: \mathbb{Z}_8 \to \mathbb{Z}_2 \times \mathbb{Z}_4$$

$$f(0) = (0,0)$$

$f(1) = (a, b)$

$\mathrm{red}(1) = 8 \Rightarrow$ TUDI (a,b) MORA BITI REDA 8

$$4 \cdot (a,b) = (4a \pmod 2,\ 4b \pmod 4)$$
$$= (0, 0)$$

$\Rightarrow$ V $\mathbb{Z}_2 \times \mathbb{Z}_4$ VSI EL. NAJVEČ REDA 4

$\Rightarrow$ NI EL. REDA 8 $\Rightarrow$ NISTA IZOMORFNA

SPLOŠNO:

$$\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \cong \mathbb{Z}_{n_1 \cdot n_2}$$
$$\Updownarrow$$
n_1 in n_2 TUJI

$$\mathrm{red}((1,1)) = \mathrm{lcm}(n_1, n_2)$$

3. $(\mathbb{Z}_{12}^*, \cdot)$

$\cdot$	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

GRUPA IMA 4 EL. $\Rightarrow$ IZOMORFNA LAHKO

$\mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2$

- ČE $\mathbb{Z}_{12}^* \cong \mathbb{Z}_4$, POTEM BI $\exists$ EL. REDA 4 V $\mathbb{Z}_{12}^*$. AMPAK $x^2 = 1$ $\forall x \in \mathbb{Z}_{12}^*$

$\rightarrow\leftarrow$

- $f : \mathbb{Z}_{12}^* \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2$

$$1 \longmapsto (0, 0)$$
$$5 \longmapsto (1, 1)$$
$$7 \longmapsto (1, 0)$$

BIJEKCIJA

$$11 \longmapsto (0,1)$$

$$f(5\cdot 7) \stackrel{?}{=} f(5) + f(7)$$

$$f(11) = (1,1) + (1,0) = (0,1) \checkmark$$

$$f(5\cdot 11) \stackrel{?}{=} f(5) + f(11) \checkmark$$

$$f(7\cdot 11) \stackrel{?}{=} f(7) + f(11) \checkmark$$

$$\underbrace{f(x\cdot x)}_{1} \stackrel{?}{=} \underbrace{f(x) + f(x)}_{(0,0)} \checkmark$$

HOMOMORFIZEM ✓

$(\mathbb{Z}_n^*, \cdot)$ → VSI EL. SO OBRNLJIVI

$(\mathbb{Z}_n, \cdot)$ KI

4. ALI $(\mathbb{Z}_{14}^*, \cdot) \overset{?}{\cong} (\mathbb{Z}_6, +)$?

$$|\mathbb{Z}_{14}^*| = \varphi(14) = 14 \cdot \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{7}\right) = $$
$$= 6$$

$14 = 2 \cdot 7$

$$(\mathbb{Z}_{14}^*, \cdot) \cong (\mathbb{Z}_6, +)$$
$$\Updownarrow$$

V $\mathbb{Z}_{14}^*$ OBSTAJA EL. REDA 6

$$\mathbb{Z}_{14}^* = \{1, 3, 5, 9, 11, 13\}$$

$3^1 = 3$

$3^2 = 9 \neq 1$

$3^3 = 13 \neq 1$

$3^4 = 11 \neq 1$

$3^5 = 5 \neq 1$ $\Rightarrow \operatorname{red}(3) = 6$

$3^6 = (3^3)^2 = 1$

$f : \mathbb{Z}_6 \to \mathbb{Z}_{14}^*$ $\quad f(x) = 3^x \pmod{14}$

JE BIJEKTIVNA, KER $red(3) = 6 = |\mathbb{Z}_{14}^*|$

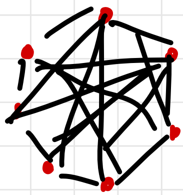
JE HOMOMORFIZEM KER

$$f(x+y) = 3^{x+y}$$
$$\shortparallel$$
$$f(x) \cdot f(y) = 3^x \cdot 3^y$$

5. NAJDI GRAF, KATEREGA GRUPA AVTOMORFIZMOV JE:

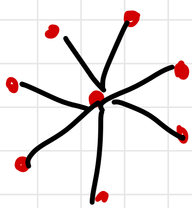
- S_n

K_n



VSAKA PERMUTACIJA AVTOMORFIZEM

$K_{1,n}$



- $\mathbb{Z}_2^k = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \dots \times \mathbb{Z}_2$

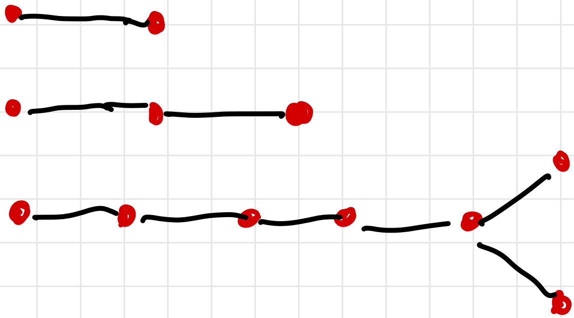
ALI Q_k ?

Q_2 

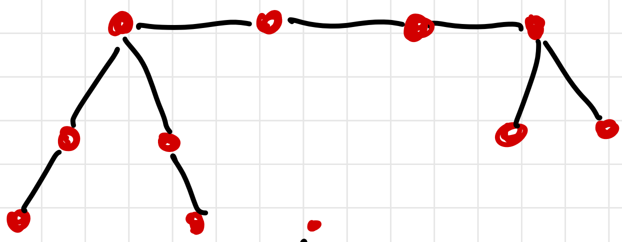
$$\mathrm{Aut}(Q_2) \overset{?}{\cong} \mathbb{Z}_2 \times \mathbb{Z}_2$$

NE KER $|\mathrm{Aut}(Q_2)| = 8$

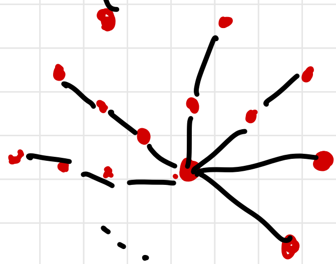
SAMO $\mathbb{Z}_2$

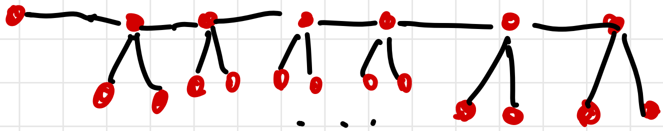


SAMO $\mathbb{Z}_2 \times \mathbb{Z}_2$



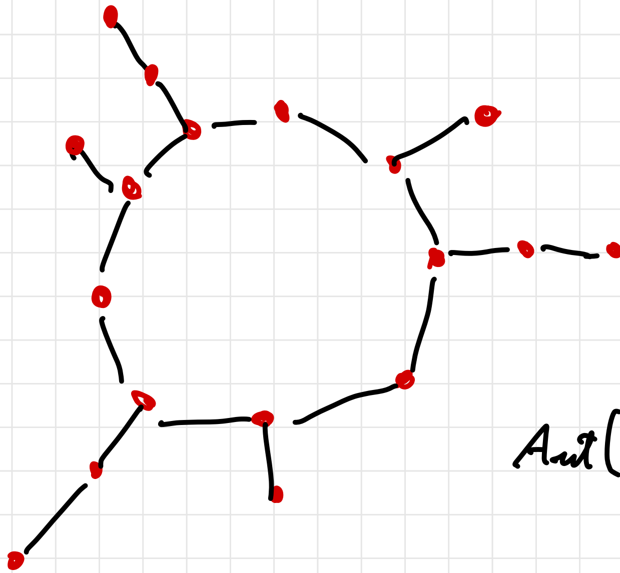
$\mathbb{Z}_2^k$



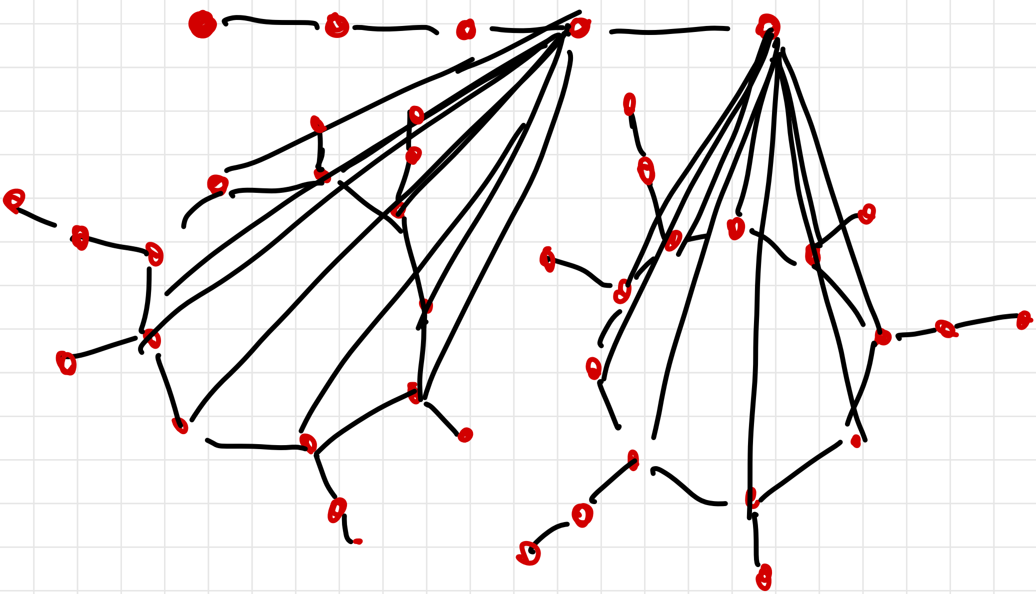


- $\mathbb{Z}_n$

  $\mathbb{Z}_n \times \mathbb{Z}_m$



$\mathrm{Aut}(G) \cong \mathbb{Z}_3$



6. $G = \langle a, b \mid a^n = e, b^m = e, ab = ba \rangle$

G VSEBUJE $abaabba \ldots ab$

VSE MOŽNE KOMBINACIJE

$|G| = ?$

$abaabbaba =$

$= aba^2b^2aba =$

$= aba^2b^2a^2b$

$= a^5b^4$

$\Rightarrow$ VSAK EL. LAHKO PREDSTAVIM KOT $a^k b^l$

KER $a^n = e$ $b^m = e$

$a^k b^l$ $0 \leq k < n$, $0 \leq l < m$

$\Rightarrow |G| = n \cdot m$

$G \cong ?$ $G \cong \mathbb{Z}_n \times \mathbb{Z}_m$

$f : \mathbb{Z}_n \times \mathbb{Z}_m \to G$

$$f(x,y) = a^x b^y$$ OČITNO BIJEKCIJA

$$f((x,y)+(z,w)) = a^{x+z} b^{y+w}$$

$$f(x,y)\cdot f(z,w) = a^x b^y a^z b^w$$

$\shortparallel \leftarrow$ KOMUTATIVNOST a IN b

7. $G = \langle a, b \mid a^4 = e,\ b^2 = a^2,\ aba = b \rangle$

a) POKAŽI, DA $bab = a$

$$bab \overset{?}{=} a$$
$$abab \overset{?}{=} a^2$$
$$bb \overset{?}{=} a^2$$
$$b^2 = a^2 \checkmark$$

b) $|G| = ?$

$$aba = b$$
$$a^3 aba = a^3 b$$
$$ba = a^3 b$$

$$a\,b\,b\,a\,a\,b\,a\,\underline{b\,a}\,b =$$

$$= a\,b\,\underline{b\,a}\,a\,\underline{b\,a}\,a^3\,b\,b =$$

$$= \ldots = a^k\, b^l$$

$\Rightarrow$ VSI EL. OBLIKE $a^k b^l$

$0 \leq k < 4$

$0 \leq l < 4$

$\Rightarrow$ KER $b^2 = a^2$

$0 \leq l < 2$

$$\Rightarrow |G| = 4 \cdot 2 = 8$$

7. $G = \langle a, b \mid a^n = b^2 = (ab)^2 = e \rangle$

$|G| = ?$

$$(ab)^2 = e$$

$$a^{-1} \rightarrow \quad ab\,ab = e \quad \leftarrow b^{-1}$$

$$ba = a^{-1} b^{-1}$$

$$ba = a^{-1} b$$

$$ba = a^{n-1}b$$

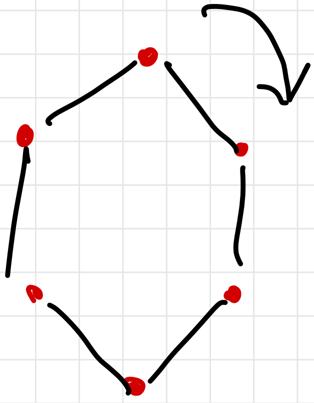
$\Rightarrow$ VSI EL. OBLIKE $a^k b^l$ $\quad 0 \leq k < n$, $\; 0 \leq l < 2$

$$|G| = 2 \cdot n$$

G POMEMBNA:

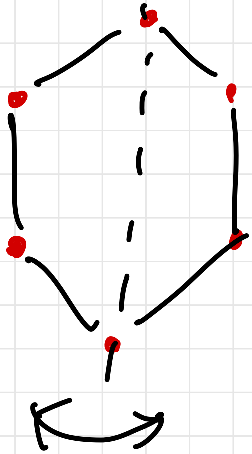
DIEDERSKA GRUPA D_{2n}

9. OPIŠI GRUPO SIMETRIJ PRAVILNEGA n-KOTNIKA Z GENERATORJI IN RELACIJAMI.



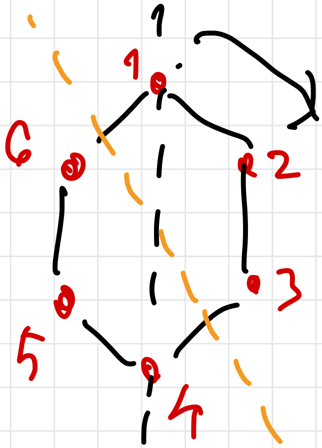
a ... ROTACIJA V DESNO ZA $\frac{2\pi}{n}$

$$a^n = e$$



b ... ZRCALJENJE

$$b^2 = e$$



$a \cdot b$:

$1 \to 6$
$2 \to 5$
$3 \to 4$
$4 \to 3$
$5 \to 2$
$6 \to 1$

$\Rightarrow$ $a \cdot b$ – ZRCALJENJE

$$(a \cdot b)^2 = e \quad \Rightarrow \quad \boxed{G \cong D_{2n}}$$