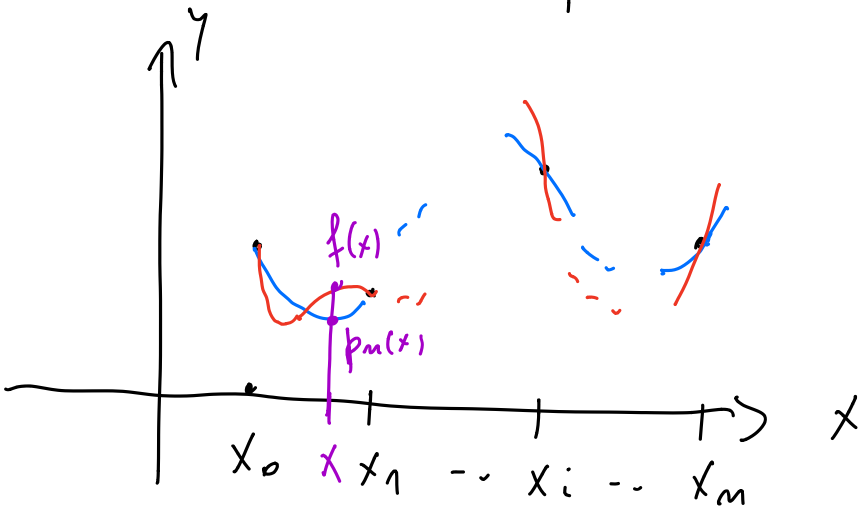
Napaka pri interpolaciji:



f – funkcija
p_n – interp. pol.

$$|f(x) - p_n(x)| = ?, \quad x \in [x_0, x_n]$$

Lema: Če je f $(n+1)$-krat zvezno odvedljiva in je p_n interpolacijski polinom, ki se z f ujema v paroma različnih točkah $x_0, x_1, \dots, x_n$, $x_i \in (a,b)$. Potem obstaja $\xi \in (a,b)$, da je

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,\omega(x),$$

$$\omega(x) = (x-x_0)\cdots(x-x_n).$$

Konvergenca $\lim_{n\to\infty} |f(x) - p_n(x)| = 0$

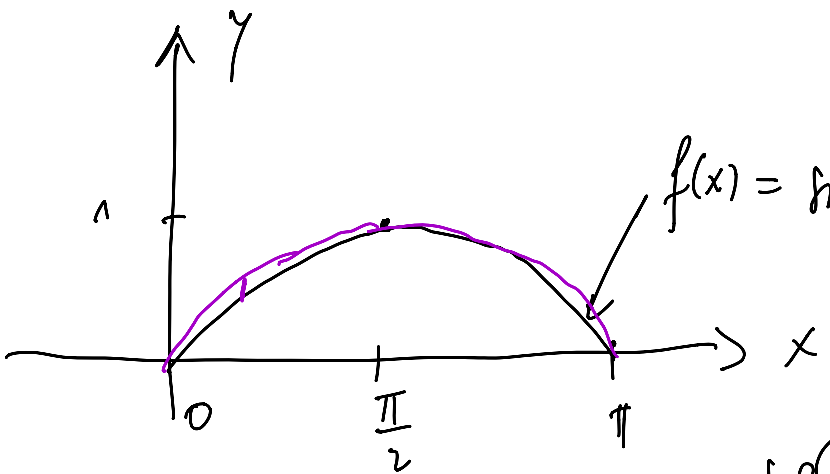
NI ZAGOTOVLJENA!

Rungejev protiprimer!!!

Primer: $f(x) = \sin x$, $x_0 = 0$, $x_1 = \frac{\pi}{2}$, $x_3 = \pi$

Kako lahko ocenimo napako

$$|f(x) - p_2(x)|, \quad x \in [0, \pi] \; ?$$



Lema: $|f(x) - p_2(x)| = \left| \frac{f^{(3)}(\xi)}{3!} (x-x_0)(x-x_1)(x-x_2) \right|$

$$\Rightarrow |f(x) - p_2(x)| \leq \max_{0 \leq \xi \leq \pi} \frac{|f^{(3)}(\xi)|}{6} \cdot \max_{0 \leq x \leq \pi} \left| x\left(x-\frac{\pi}{2}\right)(x-\pi) \right|$$

$$\max_{0 \leq \xi \leq \pi} |f^{(3)}(\xi)| \leq 1$$

$$g(x) = x\left(x - \frac{\pi}{2}\right)(x - \pi) \quad \text{na } [0, \pi].$$

$$g'(x) = \left(x - \frac{\pi}{2}\right)(x - \pi) + x(x - \pi) + x\left(x - \frac{\pi}{2}\right)$$

$$= 3x^2 - \frac{3\pi}{2}x - \pi x - \frac{\pi}{2}x + \frac{\pi^2}{2}$$

$$= 3x^2 - 3\pi x + \frac{\pi^2}{2} = 0$$

$$\tilde{x}_{1,2} = \frac{3\pi \pm \sqrt{9\pi^2 - 6\pi^2}}{6} = \frac{3\pi \pm \sqrt{3}\pi}{6}$$

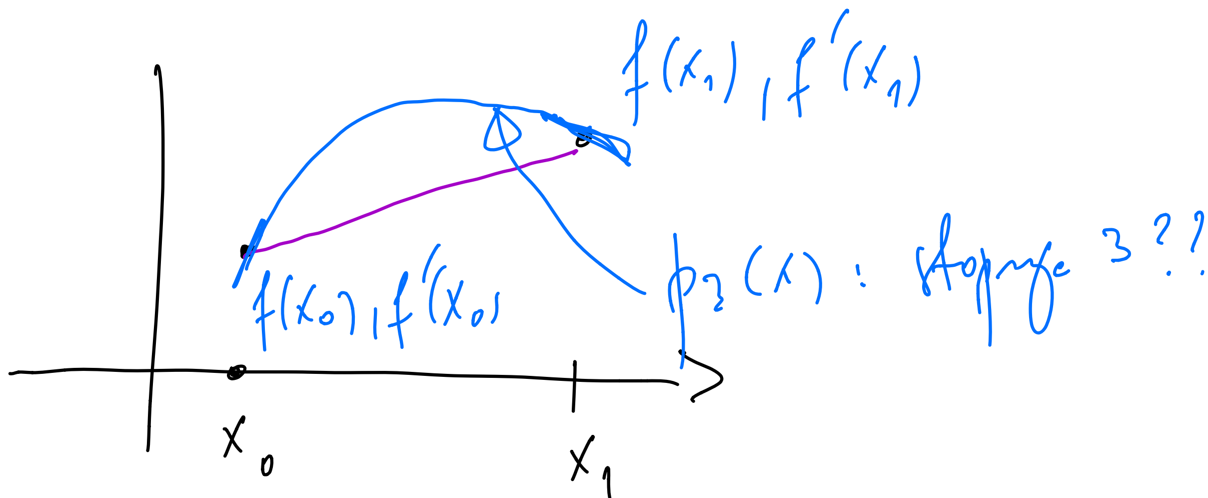
$$= \frac{\pi}{2} \pm \frac{\sqrt{3}}{6}\pi$$

$$|g(\tilde{x}_1)| = |g(\tilde{x}_2)| \doteq 1{,}49$$

$$|f(x) - p_2(x)| \le \frac{1}{6} \cdot 1{,}49 \doteq 0{,}25.$$

Kaj pa če želimo poleg vrednosti interpolirati tudi odvode?

<u>Newtonova oblika</u>



<u>Definicija</u>: Deljena diferenca funkcije f na točkah $x_0, \ldots, x_k$ je <u>vodilni koeficient</u> polinoma st. $\le k$, ki se

z f ujema v $x_0, x_1, \dots, x_k$.

Označimo jo $[x_0, x_1, \dots, x_k]f$.

Če se točke ponavljajo: $x_0 = x_1 = \dots = x_k$,

potem definiramo $[x_0, \dots, x_k] = \dfrac{f^{(k)}(x_0)}{k!}$

<u>Ideja:</u> $x_0, x_0 + h$; $h \ll 1$:

$$[x_0, x_0 + h]f = \frac{f(x_0 + h) - f(x_0)}{x_0 + h - x_0}$$

$$= \frac{f(x_0 + h) - f(x_0)}{h} \xrightarrow{h \to 0} f'(x_0)$$

Newtonova oblika interp. polinoma:

$$p_n(x) = [x_0]f + (x - x_0)[x_0, x_1]f + \dots$$

$$+ (x - x_0)(x - x_1)\dots(x - x_{n-1})[x_0, \dots, x_n]f.$$

Če se točke ponavljajo, potem p_n

interpolira tudi ustrezne vrednosti

višjih odvodov v x_i.

Deljene diference lahko računamo rekurzivno:

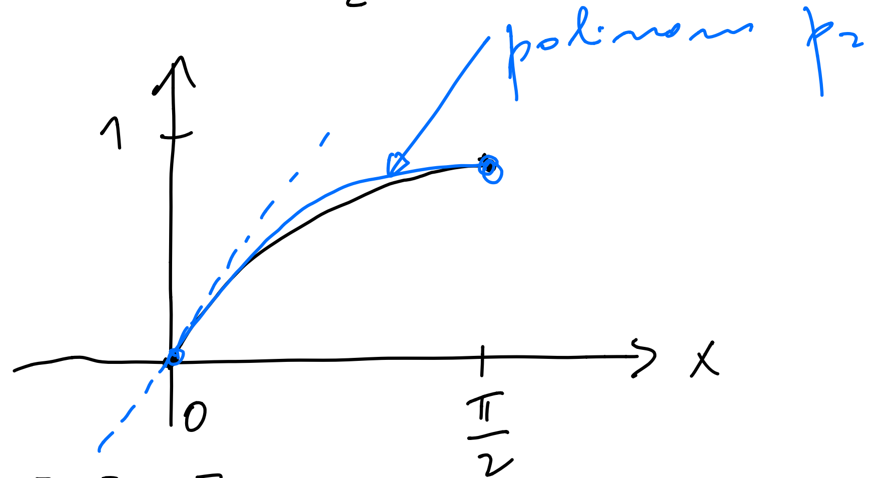
$$[x_0, x_1, \ldots, x_k]f = \begin{cases} \dfrac{f^{(k)}(x_0)}{k!}; & x_0 = x_1 = \ldots = x_k, \\[2ex] \dfrac{[x_1, \ldots, x_k]f - [x_0, \ldots, x_{k-1}]f}{x_k - x_0} \end{cases}$$

Opomba: $[x_0]f = f(x_0)$.

Napaka:

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\omega(x).$$

Primer: $f(x) = \sin x$, $x_0 = 0$, $x_1 = 0$, $x_2 = \frac{\pi}{2}$.



x_i	$[\cdot]$	$[\cdot\ \cdot]$	
0	0		
0	1	1	
$\pi/2$	1	$\dfrac{1-0}{\pi/2-0} = \dfrac{2}{\pi}$	$\dfrac{\frac{2}{\pi}-1}{\pi/2-0} = \dfrac{2}{\pi^2}(2-\pi)$

$$p_2(x) = [x_0]f + (x-x_0)[x_0,x_1]f$$

$$+ (x-x_0)(x-x_1)[x_0,x_1,x_2]f$$

$$= 0 + x\cdot 1 + x\left(x-\frac{\pi}{2}\right)\frac{2}{\pi^2}(2-\pi)$$

<u>Napaka:</u>

$$\max_{0\le x\le \frac{\pi}{2}} |f(x)-p_2(x)| \le \max_{0\le x\le \frac{\pi}{2}} \frac{|f^{(3)}(\xi)|}{6} \max_{0\le x\le\frac{\pi}{2}} \underbrace{\left|x^2\left(x-\frac{\pi}{2}\right)\right|}_{\omega(x)}$$

$\le \frac{1}{6}$

$$\omega'(x) = 2x\left(x-\frac{\pi}{2}\right)+x^2 = 0$$

$$= x(2x-\pi+x) = 0$$

$$\tilde{x}_1 = 0, \quad \tilde{x}_2 = \frac{\pi}{3}$$

$$\Rightarrow \max_{0\le x\le\frac{\pi}{2}} |\omega(x)| = \left|\omega\left(\frac{\pi}{3}\right)\right| = \left|\frac{\pi^2}{9}\left(\frac{\pi}{3}-\frac{\pi}{2}\right)\right|$$

$$= \frac{\pi^3}{9}\,\frac{1}{6} = \frac{\pi^3}{54}$$

$$\max_{0\le x\le\frac{\pi}{2}} |f(x)-p_2(x)| \le \frac{\pi^3}{6\cdot 54} = \underline{\underline{0.096}}$$

Numerično računanje odvodov

Primer: $f(x) = e^x$, $x_0 = 0$

Iščemo $f'(x_0)$ (numerično)!

Ideja:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$\approx \frac{f(x_0+h_0) - f(x_0)}{h_0}; \text{ } h_0 \text{ majhen}$$

Kako numerično računati odvod?

Izberemo točke $(x_0, f_0), (x_1, f_1), \ldots, (x_k, f_k)$

in naredimo lin. kombinacijo:

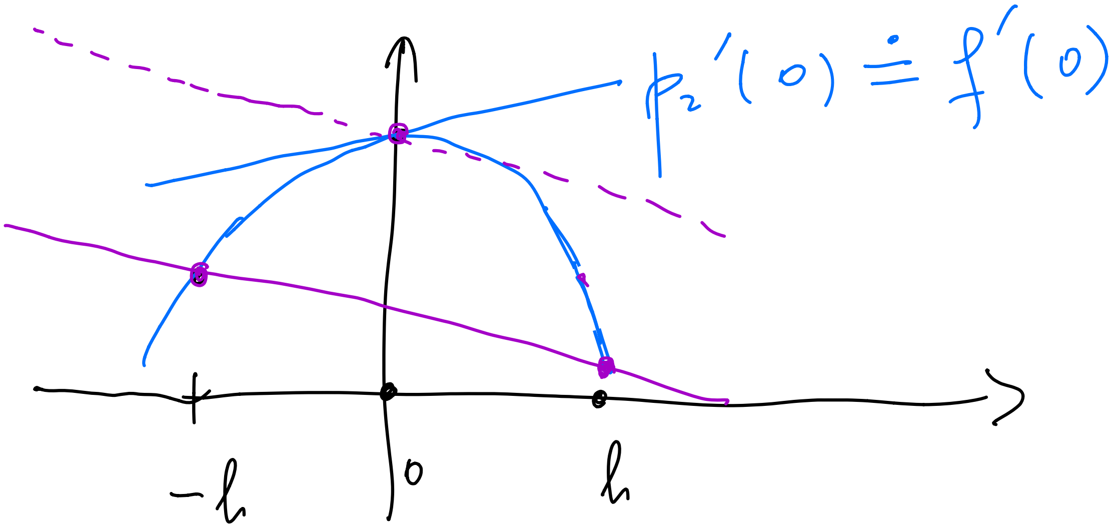
$$f'(x_j) \doteq \sum_{i=0}^{k} \alpha_i f(x_i).$$

① Vzamemo interpol. polinom st. $\leq k$ in ga odvajamo pri $x = x_j$.

② Koef. α_i določimo tako, da bo formula <u>točna</u> za polinome, <u>do</u> čim višje stopnje.

Primer: $x_0 = -h$, $x_1 = 0$, $x_2 = h$

$$f'(0) = ?$$



Uporabimo točko ②:

$$f'(0) \doteq \alpha_0 f(-h) + \alpha_1 f(0) + \alpha_2 f(h)$$

1.) $f(x) = 1$: $\quad 0 = \alpha_0 \cdot 1 + \alpha_1 \cdot 1 + \alpha_2 \cdot 1$

2) $f(x) = x$: $\quad 1 = \alpha_0 (-h) + \alpha_1 \cdot 0 + \alpha_2 \cdot h$

3) $f(x) = x^2$: $\quad 0 = \alpha_0 h^2 + \alpha_1 \cdot 0^2 + \alpha_2 h^2$

$\Downarrow$ rešimo sistem

$$\alpha_0 = -\frac{1}{2h}, \quad \alpha_1 = 0, \quad \alpha_2 = \frac{1}{2h}$$

$$f'(0) \doteq \frac{1}{2h}\left(f(h) - f(-h)\right)$$