DS 2 VAJE 13

1. $$A = \{ p + q(i\sqrt{3}) \mid p, q \in \mathbb{Q} \}$$

DOKAŽI, DA A KOLOBAR. ALI JE POLJE?

- ABELOVA GRUPA ZA $+$

$$p_1 + q_1(i\sqrt{3}) + p_2 + q_2(i\sqrt{3}) =$$
$$= (p_1 + p_2) + (q_1 + q_2) i\sqrt{3} \quad \checkmark$$

ENOTA: 0 ✓

INVERZ: $-p - q(i\sqrt{3})$ ✓

KOMUTATIVNOST ✓

- POLGRUPA ZA $\cdot$

ASOC. ✓

$$(p_1 + q_1(i\sqrt{3})) \cdot (p_2 + q_2(i\sqrt{3})) =$$
$$= \underbrace{p_1 \cdot p_2 - q_1 \cdot q_2 \cdot 3}_{\in \mathbb{Q}} + \underbrace{(p_1 \cdot q_2 + p_2 \cdot q_1)}_{\in \mathbb{Q}} i\sqrt{3} \quad \checkmark$$

- DISTRIBUTIVNOST ✓ $\Rightarrow$ JE KOLOBAR

POLJE ?

A-{0} JE AB. GRUPA $\left\{\begin{array}{l}\text{KOMUTATIVNOST} \quad \cdot \quad \checkmark \\ \text{ENOTA} \quad \cdot \ : \quad 1 = (1 + 0 \cdot i\sqrt{3}) \quad \checkmark \\ \text{INVERZ ZA} \ \cdot \ (\text{RAZEN } 0)\end{array}\right.$

ČE $p_2 + q_2 i\sqrt{3}$ INVERZ

$p_1 + q_1 i\sqrt{3}$, POTEM

$$p_1 p_2 - q_1 q_2 3 + (p_1 q_2 + p_2 q_1)\sqrt{3} i = 1$$

$$p_1 q_2 + p_2 q_1 = 0 \qquad p_2 = -\frac{p_1 q_2}{q_1}$$

$$p_1 p_2 - q_1 q_2 \cdot 3 = 1 \qquad (\text{ČE } q_1 \neq 0)$$

$$-\frac{p_1^2 q_2}{q_1} - q_1 q_2 \cdot 3 = 1$$

$$q_2 = \frac{-1}{\left(\frac{p_1^2}{q_1} + q_1 3\right)}$$

$$q_2 = \frac{-q_1}{p_1^2 + q_1^2 3}$$

OBSTAJA ČE $q_1, p_1 \neq 0$

ČE $q_1 = 0$ $\quad p_1 q_2 = 0 \Rightarrow$ ČE $p_1 \neq 0$

$\Rightarrow q_2 = 0$

$p_1 p_2 = 1$

$\Downarrow$

$p_2 = \frac{1}{p_1}$ (ČE $p_1 \neq 0$)

✓

⇒ JE POLJE

2. R MNOŽICA MATRIK OBLIKE

$$\begin{bmatrix} a & b \\ 2b & a \end{bmatrix}$$

a) $a, b \in \mathbb{R}$. DOKAŽI DA R KOLOBAR ZA OBIČAJNO SEŠT. IN MN. MATRIK.

- ABEL. GRUPA ZA +

✓ (DN)

- POLGRUPA ZA MNOŽENJE

$$\begin{bmatrix} a_1 & b_1 \\ 2b_1 & a_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ 2b_2 & a_2 \end{bmatrix} =$$

$$= \begin{bmatrix} a_1 a_2 + 2b_1 b_2 & a_1 b_2 + b_1 a_2 \\ 2b_1 a_2 + 2a_1 b_2 & 2b_1 b_2 + a_1 a_2 \end{bmatrix} ✓$$

• DISTRIB. ✓

b) $a, b \in \mathbb{Q}$. DOKAŽI, DA R OBSEG.

• KOCOBAR a) ✓

• ENOTA : $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

• INVERZI : $\begin{bmatrix} a & b \\ 2b & a \end{bmatrix}^{-1} =$

$$= \frac{1}{a^2 - 2b^2} \cdot \begin{bmatrix} a & -b \\ -2b & a \end{bmatrix}$$

ALI $a^2 - 2b^2 \neq 0$

ČE $a^2 = 2b^2$

$a = \pm\sqrt{2}\, b \leftarrow$ ČE $b \in \mathbb{Q}$

$\Rightarrow a \notin \mathbb{Q}$

$\Rightarrow$ ENAČBA NI REŠLJIVA NAD $\mathbb{Q}$

RAZEN $a = 0$
$b = 0$

$\Rightarrow$ JE OBSEG !

c) $a, b \in \mathbb{Z}_3$.

- KOLOBAR PO a)
- $$\begin{bmatrix} a & b \\ 2b & a \end{bmatrix}^{-1} =$$

$$= \left(a^2 - 2b^2\right)^{-1} \begin{bmatrix} a & -b \\ -2b & a \end{bmatrix}$$

$\Rightarrow a^2 - 2b^2 \neq 0$

ČE $a^2 = 2b^2$ (NAD $\mathbb{Z}_3$)

$$a^2 = \begin{cases} 0 & \text{ČE } a=0 \\ 1 & \text{ČE } a=1 \\ 1 & \text{ČE } a=2 \end{cases}$$

$$2b^2 = \begin{cases} 0 & \text{ČE } b=0 \\ 2 & \text{ČE } b=1 \\ 2 & \text{ČE } b=2 \end{cases}$$

ENAČBA NIMA REŠITVE, RAZEN $a=b=0$ ✓

$\Rightarrow$ R JE OBSEG

3. V $(\mathbb{Z}_{10}, +, \cdot)$ REŠI ENAČBE:

a) $3x + 5 = 2$

$3x = 7 \quad / \cdot 3^{-1}$

TUJE 3

$3 \cdot k + 10 \cdot l = 1$

REA:

	10	1	0
3	3	0	1
	1	1	-3

$k = -3$

$= 7$

$x = 7 \cdot 7 = 9$

b) $6x + 7 = 4$

$6x = 7$

6^{-1} NE OBSTAJA

KER 6 NI TUJE 10

$6x + 10k = 7$ NAD $\mathbb{Z}$

DIOFANSKA ENAČBA IMA REŠITEV

$\gcd(6, 10) \mid 7 \quad \rightarrow \leftarrow$

$\Rightarrow$ TA ENAČBA NIMA REŠITVE

c) $2x^2 + 5x + 3 = 0$

$(2x+3)(x+1) = 0$ $\Updownarrow$ EK. ZARADI DISTRIBUTIVNOSTI

- $2x+3=0$

  $2x = 7$

  $2x + 10k = 7$

  $\gcd(2,10)$ NE DELI 7

  $\Rightarrow$ NIMA REŠITVE

- $x+1=0$

  $\boxed{x = 9}$ ✓

KER IMAMO V $\mathbb{Z}_{10}$ DELITELJE NIČA:

- $2 \cdot 5 = 0$

  $2x+3 = 2$ IN $x+1 = 5$

  $\not\parallel$ $\qquad\qquad$ $\Downarrow$

  $2 \cdot 4 + 3 = 1$ $\Leftarrow$ $x = 4$

  //

- $4 \cdot 5 = 0$

  $2x+3 = 4$ IN $x+1 = 5$

  $\parallel$

  $2x+3 = 1$ $\Leftarrow$

  //

- $6 \cdot 5 = 0$ //

- $8 \cdot 5 = 0$ //

- $5 \cdot 2 = 0$

$2x + 3 = 5$ $\qquad x + 1 = 2$

$2 \cdot 1 + 3 = 5 \Leftarrow \boxed{x = 1}$

- $5 \cdot 4 = 0$

$2x + 3 = 5$ $\qquad x + 1 = 4$

$x = 3$ //

- $5 \cdot 6 = 0$

$2x + 3 = 5$ $\qquad x + 1 = 6$

$x = 5$ //

- $5 \cdot 8 = 0$

$2x + 3 = 5$ $\qquad x + 1 = 8$

$x = 7$ //

4. $(\mathbb{Z}_{12}, +, \cdot)$

a)

$$2x + 7y = 11$$

$$5x + 3y = 11 \qquad / \cdot 2$$

$$10x + 6y = 10$$

$$y = 9$$

$$\Rightarrow \quad 5x + 3 \cdot 9 = 11$$

$$5x = 8 \qquad / \cdot 5^{-1}$$

REA:

$$\begin{array}{cccc} & 12 & 1 & 0 \\ 2 & 5 & 0 & 1 \\ 2 & 2 & 1 & -2 \\ & 1 & -2 & \boxed{5} \end{array} \quad \leftarrow \quad 5 \cdot k + 12 l = 1$$

$$x = 8 \cdot 5 = 4$$

PREVERIMO

$$2 \cdot 4 + 7 \cdot 9 = 11 \quad \checkmark$$

$$5 \cdot 4 + 3 \cdot 9 = 11 \quad \checkmark$$

ZAKAJ PREVERIMO? KER SMO MNOŽILI Z 2 (NEOBRNLJIV KORAK)

PRIMER

$0 \cdot x = 6$ ← NIMA REŠITVE

$\downarrow \cdot 2$

$0 \cdot x = 2 \cdot 6 = 12 = 0$ ← IMA REŠITVE

b) $3x + 7y = 4$

$6x + y = 1$

$\cdot 2$

$+$

$6x + 2y = 8$

$3y = 9$

$3y + 12k = 9 \quad / : \gcd(3,12)$

$y + 4k = 3$

$y = 3 - 4k$

$y = 3, 7, 11$

- $y = 3$

$3x + 9 = 4$

$3x = 7$

$3x + 12k = 7$ //

- $y = 7$ $\quad 3x + 1 = 4$

$3x = 3$

$3x + 12y = 3$

$x + 4y = 1$

$x = 1 - 4y$

$x = 1, 5, 9$

$3 \cdot 1 + 7 \cdot 7 = 4$ ✓

$6 \cdot 1 + 7 = 1$ ✓

$y = 7 \quad x = 1$ ✓

$3 \cdot 5 + 7 \cdot 7 = 4$ ✓

$6 \cdot 5 + 7 = 1$ ✓

$y = 7 \quad x = 5$ ✓

$3 \cdot 9 + 7 \cdot 7 = 4$ ✓

$6 \cdot 9 + 7 = 1$ ✓

$y = 7 \quad x = 9$ ✓

- $y = 11$ $\quad 3 \cdot x + 7 \cdot 11 = 4$

$3x = 11$

$3x + 12y = 11$

//

5. $(\mathbb{Z}_{103}, +, \cdot)$

$$30x + 7y + 16z = 4$$
$$6x + y + 55z = 1$$
$$x + y + 13z = 22$$

$$\begin{bmatrix} 30 & 7 & 16 \\ 6 & 1 & 55 \\ 1 & 1 & 13 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 22 \end{bmatrix}$$

$\parallel$

A

NAD $\mathbb{Z}_{103}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} 4 \\ 1 \\ 22 \end{bmatrix}$$

$$A^{-1} = (\det A)^{-1} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$$A^{-1} = \Big(30\cdot(1\cdot 13 - 55\cdot 1) - 7\cdot(6\cdot 13 - 55\cdot 1) + 16\cdot\big(6\cdot 1 - 11\big)\Big)^{-1} \cdot \begin{bmatrix} & & \end{bmatrix}$$

$$= 3^{-1} \begin{bmatrix} 61 & 29 & 59 \\ 80 & 64 & 100 \\ 5 & 80 & 91 \end{bmatrix} =$$

REA:

103		1	0	
34	3	0	1	
	1	1	$\boxed{-34}$	

$3^{-1} = 69$

$$A^{-1} = \begin{bmatrix} 89 & 44 & 54 \\ 61 & 90 & 102 \\ 36 & 61 & 99 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} 4 \\ 1 \\ 22 \end{bmatrix} = \begin{bmatrix} 43 \\ 3 \\ 14 \end{bmatrix}$$