Algoritem ALS-CP za iskanje najboljše aproksimacije s tenzorjem ranga r z uporabo alternirajočih najmanjših kvadratov

(izpeljava za tenzor reda 3)

$$\overset{\mathbb{R}^{m_1\times m_2\times m_3}}{\overset{\rotatebox{90}{$\in$}}{\mathcal{X}}} \approx \sum_{i=1}^{r} \overset{\mathbb{R}}{\overset{\rotatebox{90}{$\in$}}{\lambda_i}}\; \overset{\mathbb{R}^{m_1}}{\overset{\rotatebox{90}{$\in$}}{a_i}} \circ \overset{\mathbb{R}^{m_2}}{\overset{\rotatebox{90}{$\in$}}{b_i}} \circ \overset{\mathbb{R}^{m_3}}{\overset{\rotatebox{90}{$\in$}}{c_i}} = [\![\lambda ; A, B, C]\!]$$

normirani

$$A = [a_1 \ldots a_r], \quad B = [b_1 \ldots b_r], \quad C = [c_1 \ldots c_r], \quad \lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_r \end{bmatrix}$$

<u>osnovna iteracija algoritma</u>

izberi A_0, B_0, C_0

$k = 0, 1, \ldots$

- poišči A_{k+1}, ki minimizira $\| \mathcal{X} - [\![A_{k+1}, B_k, C_k]\!] \|$, nato normiraj stolpce tenzorja A_{k+1}
- poišči B_{k+1}, -||- $\| \mathcal{X} - [\![A_{k+1}, B_{k+1}, C_k]\!] \|$,
  -||- B_{k+1}
- poišči S, -||- $\| \mathcal{X} - [\![A_{k+1}, B_{k+1}, S]\!] \|$,
  nato $S = \mathrm{diag}(\lambda)\, C_{k+1}$, kjer ima C_{k+1} normirane stolpce

<u>LS minimizacija znotraj algoritma</u>

BŠS: iščemo B, ki minimizira $\| \mathcal{X} - \overbrace{[\![A, B, C]\!]}^{=:\mathcal{Y}} \| = \| \mathcal{X}_{(2)} - \mathcal{Y}_{(2)} \|_F$

$$\mathcal{Y} = \sum_{i=1}^{r} a_i \circ b_i \circ c_i \Rightarrow \mathcal{Y}_{(2)} = \sum_{i=1}^{r} b_i (c_i \otimes a_i)^T$$

$$= [b_1 \cdots b_r] \begin{bmatrix} (c_1 \otimes a_1)^T \\ \vdots \\ (c_r \otimes a_r)^T \end{bmatrix}$$

$$= B\, (C \odot A)^T$$

Khatri-Raojev produkt

$$\min_B \| \mathcal{X}_{(2)} - B(C \odot A)^T \|_F = \min_B \| (C \odot A) B^T - \mathcal{X}_{(2)}^T \|_F$$

predoločen sistem
za neznano matriko B^T

normalni sistem:

$$\underbrace{(C \odot A)^T (C \odot A)}_{(*)} B^T = (C \odot A)^T \mathcal{X}_{(2)}^T$$

$$C \odot A = [\, c_1 \otimes a_1 \ \cdots \ c_r \otimes a_r \,]$$

$$(*) = (C \odot A)^T (C \odot A) = \begin{bmatrix} c_1^T \otimes a_1^T \\ \vdots \\ c_r^T \otimes a_r^T \end{bmatrix} [\, c_1 \otimes a_1 \ \cdots \ c_r \otimes a_r \,]$$

$$= \begin{bmatrix} (c_1^T \otimes a_1^T)(c_1 \otimes a_1) & \cdots & (c_1^T \otimes a_1^T)(c_r \otimes a_r) \\ \vdots & & \vdots \\ (c_r^T \otimes a_r^T)(c_1 \otimes a_1) & \cdots & (c_r^T \otimes a_r^T)(c_r \otimes a_r) \end{bmatrix}$$

"$(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$"

$$= \begin{bmatrix} (c_1^T c_1) \otimes (a_1^T a_1) & \cdots & (c_1^T c_r) \otimes (a_1^T a_r) \\ \vdots & & \vdots \\ (c_r^T c_1) \otimes (a_r^T a_1) & \cdots & (c_r^T c_r) \otimes (a_r^T a_r) \end{bmatrix}$$

$$= \begin{bmatrix} (c_1^T c_1) \cdot (a_1^T a_1) & \cdots & (c_1^T c_r) \cdot (a_1^T a_r) \\ \vdots & & \vdots \\ (c_r^T c_1) \cdot (a_r^T a_1) & \cdots & (c_r^T c_r) \cdot (a_r^T a_r) \end{bmatrix}$$

Hadamardov produkt

$$= \begin{bmatrix} c_1^T c_1 & \cdots & c_1^T c_r \\ \vdots & & \\ c_r^T c_1 & \cdots & c_r^T c_r \end{bmatrix} * \begin{bmatrix} a_1^T a_1 & \cdots & a_1^T a_r \\ \vdots & & \vdots \\ a_r^T a_1 & \cdots & a_r^T a_r \end{bmatrix}$$

$$= \begin{bmatrix} c_1^T \\ \vdots \\ c_r^T \end{bmatrix} [\, c_1 \cdots c_r \,] * \begin{bmatrix} a_1^T \\ \vdots \\ a_r^T \end{bmatrix} [\, a_1 \cdots a_r \,]$$

$$= \underline{(C^T C) * (A^T A)}$$