Reševanje sistemov dif. enačb 1. reda

$$y_1' = f_1(x, y_1, \dots, y_d)$$
$$y_2' = f_2(x, y_1, \dots, y_d)$$
$$\vdots$$
$$y_d' = f_d(x, y_1, \dots, y_d)$$

Začetni pogoji: $y_1(a) = y_{1a}, \dots, y_d(a) = y_{da}$

Zapis v vektorski obliki:

$$Y = (y_1, y_2, \dots, y_d)^T, \quad F = (f_1, f_2, \dots, f_d)^T$$
$$Y' = F(x, Y), \quad Y(a) = Y_a$$

<u>Eulerjeva metoda</u>:

$$Y_{n+1} = Y_n + h F(x_n, Y_n); \quad Y_0 = Y_a$$

<u>Primer</u>:

$$y_1' = y_2$$
$$y_2' = -y_1$$

$y_1(0) = 0, \quad y_2(0) = 1, \quad x \in [0, 2\pi].$

Analitični rešitvi: $y_1(x) = \sin x$

$y_2(x) = \cos x$

Začetni problemi višjega reda

$$y^{(k)} = f(x, y, y', \dots, y^{(k-1)})$$

$$y(a) = y_a, \quad y'(a) = y_a', \quad \dots, \quad y^{(k-1)}(a) = y_a^{(k-1)}$$

Uvedemo nove spremenljivke:

$$y_1 = y, \quad y_2 = y', \quad \dots, \quad y_k = y^{(k-1)}$$

$$y_1' = y_2$$

$$y_2' = y_3$$

$$\vdots$$

$$y_{k-1}' = y_k$$

$$y_k' = f(x, y_1, y_2, \dots, y_k)$$

$$y_1(a) = y_a, \quad y_2(a) = y_a', \quad \dots, \quad y_k(a) = y_a^{(k-1)}$$

Primer: $y'' = -y$; $y(0)=0$, $y'(0)=1$

$y_1 = y$, $y_2 = y'$ $\Rightarrow$

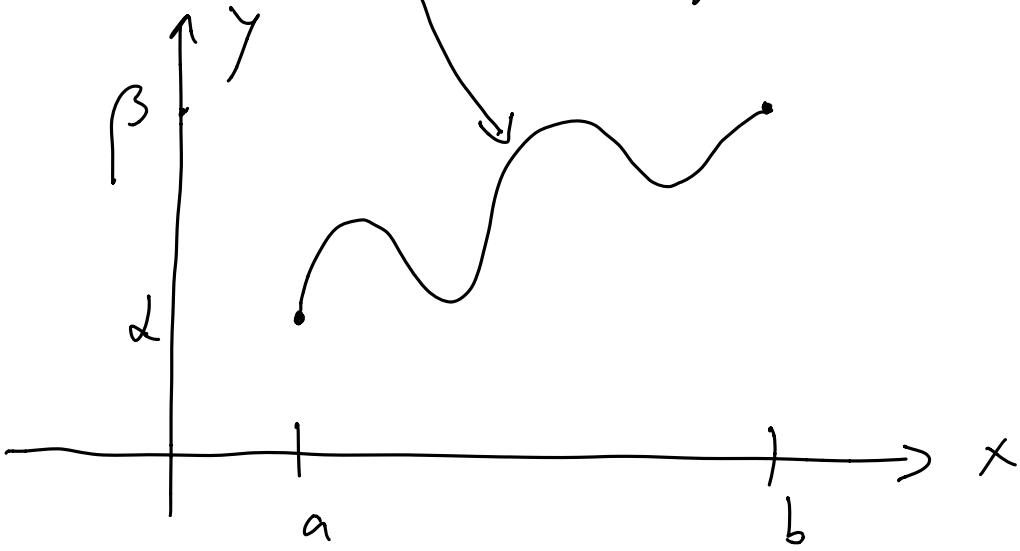
$$\boxed{y_1' = y_2, \quad y_2' = -y_1}$$

Primer: $y'' = xy$, $y(0) = y_0$, $y'(0) = y_0'$

Airyjeva funkcija

Robni problemi drugega reda

$$y'' = f(x, y, y'); \quad y(a) = \alpha, \quad y(b) = \beta$$



· Prevedemo na sistem 1. reda:

$y_1 = y$, $y_2 = y' \Rightarrow$

$$\boxed{\begin{aligned} y_1' &= y_2 \\ y_2' &= f(x, y_1, y_2) \\ y_1(a) &= \alpha, \quad \boxed{y_1(b) = \beta} \end{aligned}}$$

Linearni robni problem drugega reda

$$-y'' + py' + qy = r, \qquad (*)$$

kjer so p, q, r funkcije x (neodvisne spremenljivke)

Če sta $y^{(1)}$ in $y^{(2)}$ rešitvi $(*)$, je tudi $y = \lambda y^{(1)} + (1-\lambda) y^{(2)}$, $\lambda \in \mathbb{R}$, rešitev $(*)$.

Res:

$$-y'' + py' + qy = -\left(\lambda y^{(1)} + (1-\lambda) y^{(2)}\right)''$$

$$+ p\left(\lambda y^{(1)} + (1-\lambda) y^{(2)}\right)' + q\left(\lambda y^{(1)} + (1-\lambda) y^{(2)}\right)$$

$$= -\lambda y^{(1)''} - (1-\lambda) y^{(2)''} + \lambda p y^{(1)'} + (1-\lambda) p y^{(2)'}$$

$$+ \lambda q y^{(1)} + (1-\lambda) q y^{(2)}$$

$$= -\underline{\lambda y^{(1)''} + \lambda p y^{(1)'} + \lambda q y^{(1)}}$$

$$-(1-\lambda) y^{(2)''} + (1-\lambda) p y^{(2)'} + (1-\lambda) q y^{(2)}$$

$$= \lambda(\quad r \quad) + (1-\lambda)(\quad r \quad)$$

$$= \lambda r + r - \lambda r = r \quad \checkmark$$

Enako velja za začetni pogoj

$y^{(1)}(a) = \alpha$ in $y^{(2)}(a) = \alpha \Rightarrow$

$y(a) = \alpha$: $y(a) = \lambda y^{(1)}(a) + (1-\lambda) y^{(2)}(a)$

$$= \lambda\alpha + (1-\lambda)\alpha = \alpha \quad \checkmark$$

Kaj pa $y(b) = \lambda y^{(1)}(b) + (1-\lambda) y^{(2)}(b)$

$\neq \beta$
v splošnem

Lahko pa zahtevamo, da je

$$\lambda y^{(1)}(b) + (1-\lambda) y^{(2)}(b) \overset{!!}{=} \beta$$

$$\boxed{\lambda = \frac{\beta - y^{(2)}(b)}{y^{(1)}(b) - y^{(2)}(b)}}$$

Ideja reševanja:

- Rešimo začetni problem

$$-y'' + p y' + q y = r, \quad \begin{aligned} y(a) &= \alpha \\ y'(a) &= \delta_i \end{aligned}$$

za $i = 1, 2$; $\delta_1 \neq \delta_2$.

- Dobimo rešitvi $y^{(1)}$ in $y^{(2)}$
- Določimo λ:

$$\lambda = \frac{\beta - y^{(2)}(b)}{y^{(1)}(b) - y^{(2)}(b)}$$

- Konstruiramo rešitev:

$$y = \lambda y^{(1)} + (1-\lambda) y^{(2)}.$$

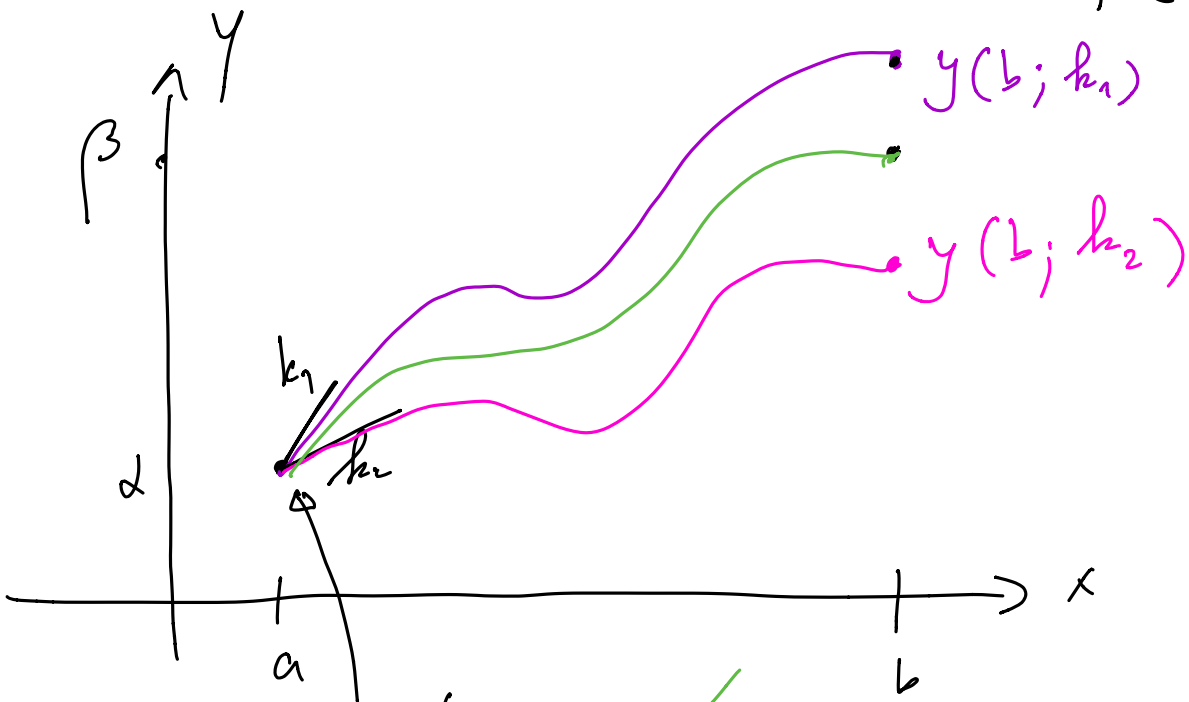
<u>Primer</u>: $-y'' - y = 0$

$y(0) = 0 \qquad y(\frac{\pi}{2}) = 1$

Analitična rešitev je $y = \sin x$.

Streljna metoda

Kaj če je dif. enačba, torej funkcija f, nelinearna? Npr: $y'' = \sin(xy) + \cos(y')$



$y(a) = \alpha$ ✓

$y'(a) = ?$

Ideja: • izberemo $y'(a) = k_1$
$\Rightarrow$ dobimo rešitev $y(x; k_1)$

• izberemo $y'(a) = k_2$ $\Rightarrow$

dobimo rešitev $y(x; k_2)$

• ugibamo naprej, ali
pa uporabimo malo matematike

$$F(k) = y(b; k) - \beta$$

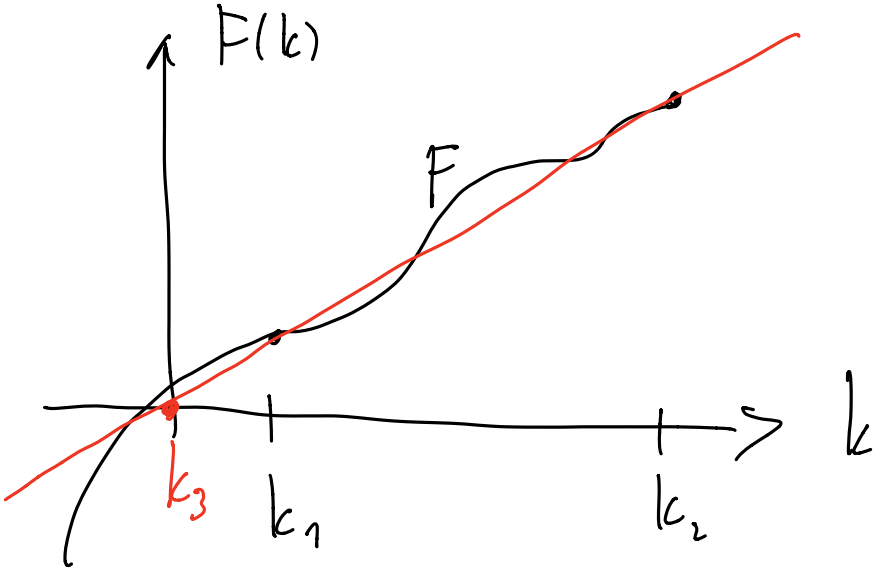
Iščemo ničlo F!!!

• bisekcija: DA, če je $F(k_1) \cdot F(k_2) < 0$

• navadna iteracija: $F(k) = 0$
$\Rightarrow k = G(k)$??

• Newton: $F'(k) =$?

• sekantna metoda ✓



$$k_{r+2} = k_{r+1} - \frac{F(k_{r+1})\,(k_{r+1} - k_r)}{F(k_{r+1}) - F(k_r)}\,;\; r = 1, 2, \ldots$$