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3. izpit iz Algebre 1 za finančnike

3. 9. 2021

Veliko uspeha!

Sedež (3.04)

Ime in priimek

Vpisna številka

1. naloga

Dane so točke $A(1,1,3)$, $B(-1,0,1)$, $C(1,1,-5)$ in $D(2,-1,-5)$.

a) Določi enačbo ravnine, sestavljene iz točk, ki so enako oddaljene od premic AB in CD.

b) Prezrcali premico CD preko ravnine ABC.

a) Da bo ravnina enako oddaljena od daljic AB in CD, mora ravnina biti vzporedna premicama AB in CD. Torej normala ravnine kaže v smer vektorja $\vec{AB} \times \vec{CD}$:

$\vec{AB} = (-2,-1,-2)$

$\vec{CD} = (1,-2,0)$

$\vec{AB} \times \vec{CD} = (-4,-2,5) =: \vec{m}$

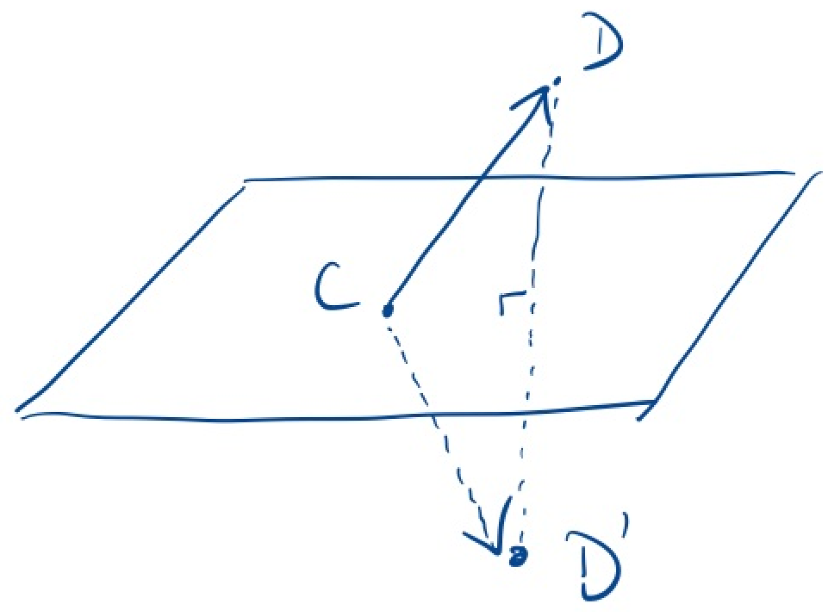
Točka, ki leži na iskani ravnini je na primer razpolovišče daljice AC, torej $T(1,1,-1)$

Enačba iskane ravnine je tako $\vec{n} \cdot (\vec{r}_T - (x,y,z)) = 0$

$$-11 + 4x + 2y - 5z = 0$$

$$4x + 2y - 5z - 11 = 0$$

b)



Naj bo D' zrcalna slika točke D preko ravnine ABC. Tedaj je $\vec{CD'} = \vec{CD} + \vec{DD'} = \vec{CD} - 2\,\mathrm{proj}_{\vec{m}}\vec{CD}$, kjer je $\vec{m}$ normala ravnine ABC, ki kaže v smer točke D, t.j. $\vec{m} \cdot \vec{CD} > 0$.

Normala $\vec{m}$: $\vec{CA} \times \vec{CB} = (0,0,8) \times (-2,-1,6) = (8,-16,0) = 8(1,-2,0) = 8\,\vec{CD}$.

Torej je premica CD pravokotna na ravnino ABC in zato je njena zrcalna slika kar premica CD.

2. naloga

Matrika $A_n = [a_{i,j}]_{i,j=1}^n$ je podana s predpisi: $a_{i,i} = a$ za $1 \leq i \leq n$, $a_{1,i} = a_{i,1} = b$ za $2 \leq i \leq n-1$, $a_{n,i} = a_{i,n} = b$ za $2 \leq i \leq n-1$ in $a_{i,j} = 0$ sicer.

a) Za vsak $n \geq 2$ izračunaj determinanto matrike A_n.

b) V odvisnosti od a in b reši sistem enačb $A_5 x = [a, a, a, a, a]^T$.

a) Če je $a=0$, sta prva in zadnja vrstica enaki, zato je $\underline{\underline{\det A_n = 0}}$.

Naj bo $a \neq 0$:

$$\det A_n = \begin{vmatrix} a & b & b & b & \cdots & b & 0 \\ b & a & 0 & 0 & \cdots & 0 & b \\ b & 0 & a & 0 & \cdots & 0 & b \\ \vdots & & & & & & \\ b & 0 & 0 & 0 & \cdots & a & b \\ 0 & b & b & b & & b & a \end{vmatrix} = \begin{vmatrix} a & b & b & b & \cdots & b & 0 \\ b & a & 0 & 0 & \cdots & 0 & b \\ 0 & -a & a & 0 & \cdots & 0 & 0 \\ \vdots & & & & & & \\ 0 & -a & 0 & 0 & \cdots & a & 0 \\ -a & 0 & 0 & 0 & \cdots & 0 & a \end{vmatrix} = a^{n-2} \begin{vmatrix} a & b & b & b & \cdots & b & 0 \\ b & a & 0 & 0 & \cdots & 0 & b \\ 0 & -1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & & \\ 0 & -1 & 0 & 0 & \cdots & 1 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 1 \end{vmatrix}$$

razvoj po 1. stolpcu

$$= a^{n-2} \begin{vmatrix} 0 & b & b & b & \cdots & b & a \\ 0 & a & 0 & 0 & \cdots & 0 & 2b \\ 0 & -1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & & \\ 0 & -1 & 0 & 0 & \cdots & 1 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 1 \end{vmatrix} = (-1)(-1)^{n+1} a^{n-1} \begin{vmatrix} b & b & b & \cdots & b & a \\ 1 & 0 & 0 & \cdots & 0 & \frac{2b}{a} \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & \cdots & 1 & 0 \end{vmatrix} = (-1)^n a^{n-1} \begin{vmatrix} b & b & b & \cdots & b & a \\ 1 & 0 & 0 & \cdots & 0 & \frac{2b}{a} \\ 0 & 1 & 0 & \cdots & 0 & \frac{2b}{a} \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \frac{2b}{a} \\ 0 & 0 & 0 & \cdots & 1 & \frac{2b}{a} \end{vmatrix}$$

$$= (-1)^n a^{n-1} \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & a - \frac{2(n-2)b^2}{a} \\ 1 & 0 & 0 & \cdots & 0 & \frac{2b}{a} \\ 0 & 1 & 0 & \cdots & 0 & \frac{2b}{a} \\ \vdots & & & & & \\ 0 & 0 & 0 & & 0 & \frac{2b}{a} \\ 0 & 0 & 0 & \cdots & 1 & \frac{2b}{a} \end{vmatrix} = (-1)^n a^{n-1} (-1)^n \left(a - \frac{2(n-2)b^2}{a}\right) \begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{vmatrix} = \underline{\underline{a^{n-1}\left(a - \frac{2(n-2)b^2}{a}\right)}}$$

b) Sistem rešimo s pomočjo razširjene matrike in Gaussove metode:

$$\tilde{A} = \left[\begin{array}{ccccc|c} a & b & b & b & 0 & a \\ b & a & 0 & 0 & b & a \\ b & 0 & a & 0 & b & a \\ b & 0 & 0 & a & b & a \\ 0 & b & b & b & a & a \end{array}\right] \sim \left[\begin{array}{ccccc|c} a & b & b & b & 0 & a \\ b & a & 0 & 0 & b & a \\ 0 & -a & a & 0 & 0 & 0 \\ 0 & -a & 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 & a & 0 \end{array}\right]$$

1) $a = b = 0$: Vsi $x \in \mathbb{R}^5$ rešijo enačbo.

2) $a = 0,\ b \neq 0$:

$$\tilde{A} \sim \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & \end{array} \\ \left[\begin{array}{cc|ccc|c} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

parametri

$$\underline{\underline{x_1 = -x_5 \qquad x_2 = -x_3 - x_4, \quad x_2, x_3, x_4 \in \mathbb{R}}}$$

3) $a \neq 0$:

$$\tilde{A} \sim \left[\begin{array}{ccccc|c} a & b & b & b & 0 & a \\ b & a & 0 & 0 & b & a \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 0 & b & b & b & a & a \\ 0 & a & 0 & 0 & 2b & a \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 0 & b & b & b & a & a \\ 0 & 1 & 0 & 0 & \frac{2b}{a} & 1 \\ 0 & 0 & 1 & 0 & \frac{2b}{a} & 1 \\ 0 & 0 & 0 & 1 & \frac{2b}{a} & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & a - \frac{6b^2}{a} & a - 3b \\ 0 & 1 & 0 & 0 & \frac{2b}{a} & 1 \\ 0 & 0 & 1 & 0 & \frac{2b}{a} & 1 \\ 0 & 0 & 0 & 1 & \frac{2b}{a} & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

3.1) $a - \frac{6b^2}{a} = 0$: $\underline{\underline{\text{sistem nima rešitve}}}$

3.2) $a - \frac{6b^2}{a} \neq 0$: $x_5 = \frac{(a-3b)a}{a^2 - 6b^2} = x_1$

$$\underline{\underline{x_2 = x_3 = x_4 = 1 - \frac{2b}{a} x_5 = 1 - \frac{2b}{\cancel{a}} \frac{(a-3b)\cancel{a}}{a^2 - 6b^2} = 1 - \frac{2b(a-3b)}{a^2 - 6b^2}}}$$

3. naloga

Prostor $\mathbb{R}^3$ je opremljen s standardnim skalarnim produktom. Ena od lastnih vrednosti sebi adjungirane linearne preslikave $A\colon \mathbb{R}^3 \to \mathbb{R}^3$ je 1, poleg tega je $A(2,-2,1) = (-2,2,-1)$ in jedro matrike A je netrivialno ter je vsebovano v ravnini $x + 2y + 2z = 0$. Za vsako naravno število n zapiši matriko preslikave A^n v standardni bazi.

- $\lambda_1 = 1$ je lastna vrednost
- $A(2,-2,1) = -(2,-2,1) \Rightarrow \lambda_2 = -1$ je lastna vrednost in pripadajoči lastni vektor je $v_2 = (2,-2,1)$
- $\ker A \neq \{0\} \Rightarrow \lambda_3 = 0$

Ker je A sebi adjungirana, so lastni vektorji, ki pripadajo različnim lastnim vrednostim, paroma pravokotni. Torej je $v_3 \perp v_2$ in v_3 leži na ravnini $x+2y+2z=0$, t.j. $v_3 \perp (1,2,2)$. Torej v_3 kaže v smer $v_2 \times \overset{2\ -2\ 1}{(1,2,2)} = (-6,-3,6)$, zato $v_3 = (2,1,-2)$. Tako lastni vektor v_1 kaže v smeri $v_2 \times v_3 = (2,-2,1) \times \overset{2\ -2\ 1}{(2,1,-2)} = (3,6,6)$, torej $v_1 = (1,2,2)$

Lastne vektorje še normiramo: $u_1 = \frac{1}{3}v_1 = \frac{1}{3}(1,2,2)$, $u_2 = \frac{1}{3}v_2 = \frac{1}{3}(2,-2,1)$, $u_3 = \frac{1}{3}v_3 = \frac{1}{3}(2,1,-2)$.

V bazi ortonormirani bazi $\mathcal{B} = \{u_1, u_2, u_3\}$ je $A_{\mathcal{B}\mathcal{B}} = \begin{bmatrix} 1 & & \\ & -1 & \\ & & 0 \end{bmatrix}$ in prehodna matrika $P_{\mathcal{S}\mathcal{B}} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix}$ je ortogonalna in zato je

$P_{\mathcal{B}\mathcal{S}} = P_{\mathcal{S}\mathcal{B}}^{-1} = P_{\mathcal{S}\mathcal{B}}^{T} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix}$.

Torej je $(A^n)_{\mathcal{S}\mathcal{S}} = P_{\mathcal{S}\mathcal{B}} A_{\mathcal{B}\mathcal{B}}^n P_{\mathcal{B}\mathcal{S}}$.

Če je n liho, je $(A^n)_{\mathcal{B}\mathcal{B}} = A_{\mathcal{B}\mathcal{B}}$ in zato $(A^n)_{\mathcal{S}\mathcal{S}} = A_{\mathcal{S}\mathcal{S}} = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & & \\ & -1 & \\ & & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 1 & -2 & 0 \\ 2 & 2 & 0 \\ 2 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} -3 & 6 & 0 \\ 6 & 0 & 6 \\ 0 & 6 & 3 \end{bmatrix}$.

Če je n sodo, je $(A^n)_{\mathcal{B}\mathcal{B}} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}$ in zato $(A^n)_{\mathcal{S}\mathcal{S}} = A_{\mathcal{S}\mathcal{S}} = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix}$.

4. naloga

Dana je matrika

$$A = \begin{bmatrix} 11t+2 & -11t & -t-1 & t-3 \\ 11t & 2-11t & -t-1 & t-3 \\ -5t-3 & 5t+3 & 2-t & t \\ 1-5t & 5t-1 & -t & t+2 \end{bmatrix}.$$

Poišči njeno jordansko formo v odvisnosti od $t \in \mathbb{R}$. Za $t = 0$ poišči tudi prehodno matriko.

$$p_A(\lambda) = \det(A-\lambda I) = \begin{vmatrix} 11t+2-\lambda & -11t & -t-1 & t-3 \\ 11t & 2-11t-\lambda & -t-1 & t-3 \\ -5t-3 & 5t+3 & 2-t-\lambda & t \\ 1-5t & 5t-1 & -t & t+2-\lambda \end{vmatrix} = \begin{vmatrix} 11t+2-\lambda & -11t & -t-1 & t-3 \\ \lambda-2 & 2-\lambda & 0 & 0 \\ -5t-3 & 5t+3 & 2-t-\lambda & t \\ 4 & -4 & \lambda-2 & 2-\lambda \end{vmatrix}$$

$$= (\lambda-2)\begin{vmatrix} 11t+2-\lambda & -11t & -t-1 & t-3 \\ 1 & -1 & 0 & 0 \\ -5t-3 & 5t+3 & 2-t-\lambda & t \\ 4 & -4 & \lambda-2 & 2-\lambda \end{vmatrix} = (\lambda-2)\begin{vmatrix} 2-\lambda & 0 & -t-1 & t-3 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 2-t-\lambda & t \\ 0 & 0 & \lambda-2 & 2-\lambda \end{vmatrix}$$

$$= (\lambda-2)^2(-1)\begin{vmatrix} 2-\lambda & -t-1 & t-3 \\ 0 & 2-t-\lambda & t \\ 0 & 1 & -1 \end{vmatrix} = (\lambda-2)^3\begin{vmatrix} 2-t-\lambda & t \\ 1 & -1 \end{vmatrix} = (\lambda-2)^3\begin{vmatrix} 2-\lambda & 0 \\ 1 & -1 \end{vmatrix} = (\lambda-2)^4$$

$\lambda_{1,2,3,4} = 2$

$$A-2I = \begin{bmatrix} 11t & -11t & -t-1 & t-3 \\ 11t & -11t & -t-1 & t-3 \\ -5t-3 & 5t+3 & -t & t \\ 1-5t & 5t-1 & -t & t \end{bmatrix} \sim \begin{bmatrix} 11t & -11t & -t-1 & t-3 \\ 0 & 0 & 0 & 0 \\ -5t-3 & 5t+3 & -t & t \\ 4 & -4 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & -t-1 & t-3 \\ 0 & 0 & -t & t \\ 1 & -1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & -1 & -3 \\ 0 & 0 & -t & t \\ 1 & -1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 4t \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

1) $t \neq 0 \Rightarrow A-2I \sim \begin{bmatrix} 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix} \Rightarrow \mathrm{rang}(A-2I) = 3 \Rightarrow$ A ima le en lastni vektor $\Rightarrow$ $\underline{\underline{J(A) = \begin{bmatrix} 2 & 1 & & \\ & 2 & 1 & \\ & & 2 & 1 \\ & & & 2 \end{bmatrix}}}$

2) $t = 0 \Rightarrow A-2I \sim \begin{bmatrix} 0 & 0 & -1 & -3 \\ 1 & -1 & 0 & 0 \end{bmatrix} \Rightarrow$ A ima dva lastna vektorja, t.j. $J(A)$ ima dve kletki

$$(A-2I)^2 = \begin{bmatrix} 0 & 0 & -1 & -3 \\ 0 & 0 & -1 & -3 \\ -3 & 3 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow m_A(\lambda) = (\lambda-2)^2 \Rightarrow$$ obe kletki sta velikosti 2×2 $\Rightarrow$ $\underline{\underline{J(A) = \begin{bmatrix} 2 & 1 & & \\ & 2 & & \\ & & 2 & 1 \\ & & & 2 \end{bmatrix}}}$

Korenska vektorja: $v \in \ker(A-2I)^2 \setminus \ker(A-2I)$

na primer $v_1^{(2)} = [1,0,0,0]^T$ in $v_2^{(2)} = [0,0,1,0]^T$

$v_1^{(1)} = (A-2I)v_1^{(2)} = [0,0,-3,1]^T$

$v_2^{(1)} = (A-2I)v_2^{(2)} = [-1,-1,0,0]^T$

(Opomba: za $v_2^{(2)}$ ne smemo izbrati $[0,1,0,0]^T$, saj sta potem vektorja $(A-2I)v_1^{(2)}$ in $(A-2I)v_2^{(2)}$ linearno odvisna)

$$P = [v_1^{(1)}, v_1^{(2)}, v_2^{(1)}, v_2^{(2)}] = \underline{\underline{\begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ -3 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}}}$$