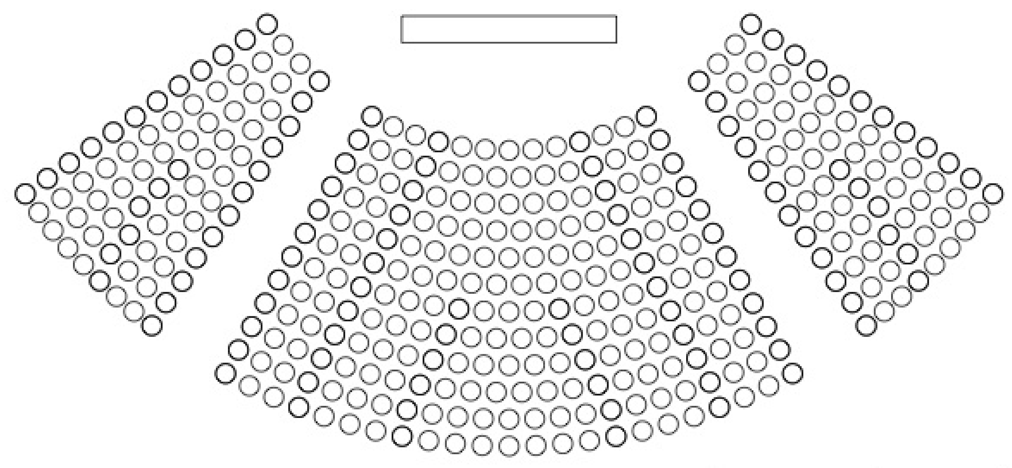
3. kolokvij iz Algebre 1 za finančnike

27. 5. 2020

Veliko uspeha!



Sedež (VFP)

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Ime in priimek

Vpisna številka

1. naloga

Naj bo $p_1(x) = x^3 + x^2$, $p_2(x) = -x^2$, $p_3(x) = x^3 + 2x^2 + x$ in $p_4(x) = 2x^3 + x^2 + 3x + 1$. Naj za linearno preslikavo $\mathcal{A}\colon \mathbb{R}_3[x] \to \mathbb{R}_3[x]$ velja $\mathcal{A}p_1 = p_3$, $\mathcal{A}p_2 = -p_3$, $\mathcal{A}p_3 = p_1 - 2p_2$ in $\mathcal{A}p_4 = -p_1 + 2p_2 + 3p_3$.

1. Zapiši matriko za $\mathcal{A}$ v standardni bazi $\mathcal{S} = \{x^3, x^2, x, 1\}$.
2. Zapiši kako ortonormirano bazo za jedro $\ker \mathcal{A}$, če je na prostoru $\mathbb{R}_3[x]$ dan skalarni produkt $\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\,dx$.

1. Naj bo $\mathcal{B} = \{p_1, p_2, p_3, p_4\}$. Matrika za A v bazi $\mathcal{B}$ je $A_{\mathcal{B}\mathcal{B}} = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -2 & 2 \\ 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Prehodna je $P_{S\mathcal{B}} = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 1 & -1 & 2 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

$$P_{\mathcal{B}S} = P_{S\mathcal{B}}^{-1}: \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & -1 & 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & -2 \\ 1 & -1 & 2 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 & 1 & -4 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$P_{\mathcal{B}S}$

$$A_{SS} = P_{S\mathcal{B}} A_{\mathcal{B}\mathcal{B}} P_{\mathcal{B}S} = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 1 & -1 & 2 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -2 & 2 \\ 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 1 & -1 & 1 & -4 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 2 & -2 & 3 & 3 \\ 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 1 & -1 & 1 & -4 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 & 4 \\ 0 & 2 & -1 & 4 \\ 0 & 1 & -2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2. Ker je $A(p_1+p_2)=0$ in $A(p_3+p_4-3p_1)=0$, sta p_1+p_2 in $p_3+p_4-3p_1$ v $\ker A$, zato $\dim \ker A \geq 2$. Ker sta p_3 in p_1-2p_2 v $\operatorname{im} A$, je $\dim \operatorname{im} A \geq 2$. Ker je $\dim \ker A + \dim \operatorname{im} A = \dim \mathbb{R}_3[x] = 4$, je $\dim \ker A = 2$ in $\{p_1+p_2, p_3+p_4-3p_1\}$ je baza za $\ker A$.

$\tilde{q}_1(x) = p_1(x) + p_2(x) = x^3 \quad \|\tilde{q}_1\|^2 = \int_{-1}^{1} x^6\,dx = \frac{2}{7} \Rightarrow q_1 = \sqrt{\frac{7}{2}}\,\tilde{q}_1, \quad \|q_1\| = 1$

$\tilde{q}_2(x) = p_3(x) + p_4(x) - 3p_1(x) = 4x+1$

liha funkcija

$\langle \tilde{q}_2, q_1\rangle = \int_{-1}^{1} \sqrt{\frac{7}{2}}\,(4x^4 + \cancel{x^3})\,dx = 4\sqrt{\frac{7}{2}}\,\frac{x^5}{5}\Big|_{-1}^{1} = \frac{8}{5}\sqrt{\frac{7}{2}}$

$\hat{q}_2(x) = \tilde{q}_2(x) - \langle \tilde{q}_2, q_1\rangle q_1 = 4x+1 - \frac{8\cdot 7}{5\cdot 2}x^3 = 4x+1-\frac{28}{5}x^3$

liha funkcija　　liha funkcija

$$\begin{aligned} \|\hat{q}_2\|^2 &= \int_{-1}^{1}\left(4x+1-\frac{28}{5}x^3\right)^2 dx = \int_{-1}^{1} 16x^2 + 1 + \frac{28^2}{25}x^6 + \cancel{8x} - \frac{8\cdot 28}{5}x^4 - \cancel{\frac{2\cdot 28}{5}x^3}\,)\,dx \\ &= 2\left(\frac{16}{3} + 1 + \frac{28^2}{25}\,\frac{1}{7} - \frac{8\cdot 28}{5}\cdot\frac{1}{5}\right) \\ &= 2\left(\frac{16}{3} + 1 + \frac{112}{25} - \frac{8(25+3)}{25}\right) \\ &= 2\left(5 + \frac{1}{3} + 1 + 4 + \frac{12}{25} - 8 - \frac{24}{25}\right) \\ &= 2\left(2 + \frac{1}{3} - \frac{12}{25}\right) = 2\left(2 + \frac{25-36}{75}\right) \\ &= 2\left(2 - \frac{11}{75}\right) = 2\,\frac{139}{75} = \frac{278}{75} \end{aligned}$$

$q_2(x) = \sqrt{\frac{75}{278}}\left(4x+1-\frac{28}{5}x^2\right)$

$\{q_1, q_2\}$ je ortonormirana baza za $\ker A$

2. naloga

Naj bo $\left\langle \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \right\rangle = x_1x_2 + y_1y_2 + 3z_1z_2 + x_1z_2 + z_1x_2 - y_1z_2 - z_1y_2$ in $A = \begin{bmatrix} 1 & 3 & -5 \\ 3 & 1 & 5 \\ 0 & 0 & 3 \end{bmatrix}$.

1. Pokaži, da je $\langle \cdot, \cdot \rangle$ skalarni produkt na $\mathbb{R}^3$.
2. Ali je A normalna glede na skalarni produkt $\langle \cdot, \cdot \rangle$?
3. Naj bo $v = (1, 1, 0)^T$. Določi A^*v glede na skalarni produkt $\langle \cdot, \cdot \rangle$.

1) $\left\langle \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \right\rangle = x_1x_2 + y_1y_2 + 3z_1z_2 + x_1z_2 + z_1x_2 - y_1z_2 - z_1y_2 = \left\langle \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \right\rangle$

$\left\langle \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}, \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} \right\rangle = (x_1+x_2)x_3 + (y_1+y_2)y_3 + 3(z_1+z_2)z_3 + (x_1+x_2)z_3 + (z_1+z_2)x_3 - (y_1+y_2)z_3 - (z_1+z_2)y_3 = \left\langle \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}, \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} \right\rangle$

$\left\langle \lambda \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \right\rangle = (\lambda x_1)x_2 + (\lambda y_1)y_2 + 3(\lambda z_1)z_2 + (\lambda x_1)z_2 + (\lambda z_1)x_2 - (\lambda y_1)z_2 - (\lambda z_1)y_2 = \lambda \left\langle \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \right\rangle$

$\left\langle \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right\rangle = x^2 + y^2 + 3z^2 + 2xz - 2yz = (x+z)^2 + (y-z)^2 + z^2 \geq 0$

$\left\langle \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right\rangle = 0 \iff x+z=0,\ y-z=0$ in $z=0 \iff x=y=z=0$

2) A je normalna $\iff$ obstaja ortogonalna baza za $\mathbb{R}^3$, ki jo sestavljajo lastni vektorji matrike A

$$p_A(\lambda) = (3-\lambda)\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = (3-\lambda)(\lambda^2 - 2\lambda + 1 - 9) = (3-\lambda)(\lambda-4)(\lambda+2)$$

$\lambda_1 = -2$: $A + 2I = \begin{bmatrix} 3 & 3 & -5 \\ 3 & 3 & 5 \\ 0 & 0 & 5 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$\lambda_2 = 3$: $A - 3I = \begin{bmatrix} -2 & 3 & -5 \\ 3 & -2 & 5 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

$\lambda_3 = 4$: $A - 4I = \begin{bmatrix} -3 & 3 & -5 \\ 3 & -3 & 5 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\langle v_1, v_2 \rangle = 1\cdot 1 + (-1)(-1) + 0 + 1\cdot(-1) + 0 - (-1)(-1) - 0 = 0 \Rightarrow v_1 \perp v_2$

$\langle v_1, v_3 \rangle = 1\cdot 1 + 1(-1) + 0 = 0 \Rightarrow v_1 \perp v_3$

$\langle v_2, v_3 \rangle = 1\cdot 1 + (-1)1 + 0 + 0 + (-1)\cdot 1 - 0 - (-1)1 = 0 \Rightarrow v_2 \perp v_3$

$\Rightarrow A$ je normalna

3) Zapišimo $A^*v = \alpha v_1 + \beta v_2 + \gamma v_3$

$$\left.\begin{aligned} \langle Av_1, v \rangle &= \langle -2v_1, v_3 \rangle = 0 \\ \langle v_1, A^*v \rangle &= \langle v_1, \alpha v_1 + \beta v_2 + \gamma v_3 \rangle = \alpha \langle v_1, v_1 \rangle \end{aligned}\right\} \Rightarrow \alpha = 0$$

$$\left.\begin{aligned} \langle Av_2, v \rangle &= \langle 3v_2, v_3 \rangle = 0 \\ \langle v_2, A^*v \rangle &= \langle v_2, \alpha v_1 + \beta v_2 + \gamma v_3 \rangle = \beta \langle v_2, v_2 \rangle \end{aligned}\right\} \Rightarrow \beta = 0$$

$$\left.\begin{aligned} \langle Av_3, v \rangle &= \langle 4v_3, v_3 \rangle = 4\langle v_3, v_3 \rangle \\ \langle v_3, A^*v \rangle &= \langle v_3, \alpha v_1 + \beta v_2 + \gamma v_3 \rangle = \gamma \langle v_3, v_3 \rangle \end{aligned}\right\} \Rightarrow \gamma = 4$$

$A^*v = 4(1,1,0)^T$

3. naloga

Katero krivuljo v ravnini prestavlja enačba $x^2 + 6\sqrt{3}xy - 5y^2 + 8 = 0$? Poišči njene glavne osi in polosi ter jo nariši.

$$A = \begin{bmatrix} 1 & 3\sqrt{3} \\ 3\sqrt{3} & -5 \end{bmatrix} \qquad p_A(\lambda) = (1-\lambda)(-5-\lambda) - 27 = \lambda^2 + 4\lambda - 32 = (\lambda+8)(\lambda-4)$$

$$\lambda_1 = 4 : \; A - 4I = \begin{bmatrix} -3 & 3\sqrt{3} \\ \cancel{3\sqrt{3}} & \cancel{-9} \end{bmatrix} \qquad u_1 = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} \qquad \|u_1\|^2 = 3+1 = 4 \Rightarrow v_1 = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\lambda_2 = -8 : \; A + 8I = \begin{bmatrix} \cancel{9} & \cancel{3\sqrt{3}} \\ 3\sqrt{3} & 3 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix} \qquad \|u_2\|^2 = 1+3 = 4 \Rightarrow v_2 = \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \Rightarrow P^{-1} = P^T = P \Rightarrow \begin{array}{l} x = \frac{\sqrt{3}}{2}x' + \frac{1}{2}y' \\ y = \frac{1}{2}x' - \frac{\sqrt{3}}{2}y' \end{array}$$

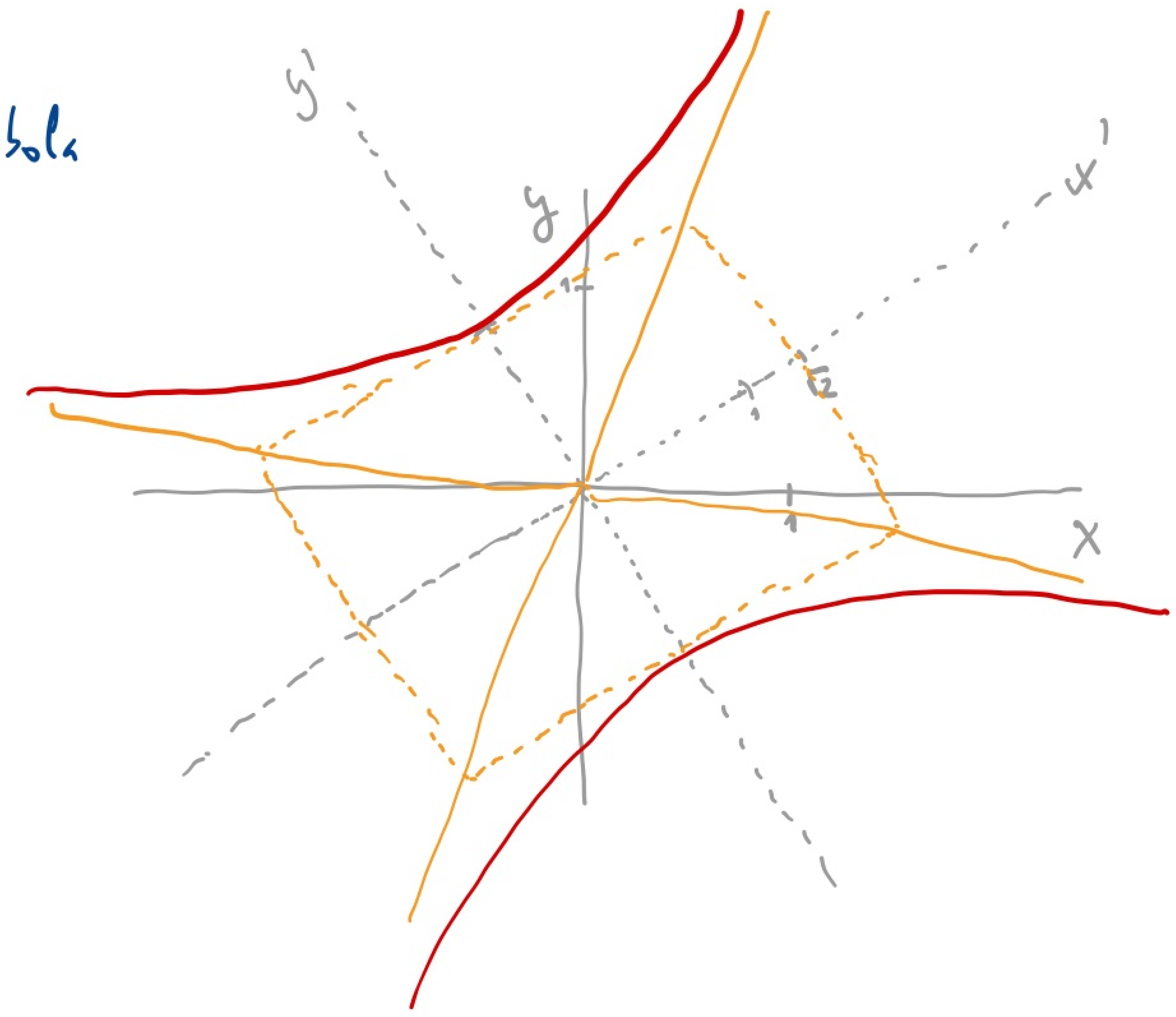
in

$x' = \frac{\sqrt{3}}{2}x + \frac{1}{2}y \Rightarrow$ y'-os $(x'=0) \Rightarrow$ premica $y = -\sqrt{3}x$

$y' = \frac{1}{2}x - \frac{\sqrt{3}}{2}y \Rightarrow$ x'-os $(y'=0) \Rightarrow$ premica $y = \frac{1}{\sqrt{3}}x$

$$4x'^2 - 8y'^2 = -8$$

$$-\frac{x'^2}{(\sqrt{2})^2} + y'^2 = 1 \qquad \text{hiperbola}$$



4. naloga

Naj bo

$$A_a = \begin{bmatrix} -2 & 0 & 1 & -1 \\ 1 & -1 & -2 & 2 \\ 0 & a+1 & -2 & 0 \\ -1 & a & 1 & -3 \end{bmatrix}$$

V odvisnosti od parametra $a \in \mathbb{R}$ določi Jordanovo kanonično formo $J(A_a)$ in pripadajočo prehodno matriko.

$$p_{A_a}(\lambda) = \begin{vmatrix} -2-\lambda & 0 & 1 & -1 \\ 1 & -1-\lambda & -2 & 2 \\ 0 & a+1 & -2-\lambda & 0 \\ -1 & a & 1 & -3-\lambda \end{vmatrix} = (-2-\lambda)\begin{vmatrix} -2-\lambda & 0 & 0 & -1 \\ 1 & -1-\lambda & 0 & 2 \\ 0 & a+1 & 1 & 0 \\ -1 & a & 1 & -3-\lambda \end{vmatrix} = (-2-\lambda)\begin{vmatrix} -2-\lambda & 0 & 0 & -1 \\ 1 & -1-\lambda & 0 & 2 \\ 0 & a+1 & 1 & 0 \\ -1 & -1 & 0 & -3-\lambda \end{vmatrix} = (-2-\lambda)\begin{vmatrix} -2-\lambda & 0 & -1 \\ 1 & -1-\lambda & 2 \\ -1 & -1 & -3-\lambda \end{vmatrix}$$

$$= (2+\lambda)^2 \begin{vmatrix} 1 & 0 & -1 \\ -1 & 1-\lambda & 2 \\ 0 & -1 & -3-\lambda \end{vmatrix} = (2+\lambda)^2 \begin{vmatrix} 1 & 0 & -1 \\ 0 & -1-\lambda & 1 \\ 0 & -1 & -3-\lambda \end{vmatrix} = (2+\lambda)^2\big((1+\lambda)(3+\lambda)+1\big) = (2+\lambda)^2(\lambda^2+4\lambda+3+1)$$

$$= (2+\lambda)^4$$

$$A_a + 2I = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 1 & 1 & -2 & 2 \\ 0 & a+1 & 0 & 0 \\ -1 & a & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & a+1 & 0 & 0 \\ -1 & a & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & a+1 & 0 & 0 \\ 0 & a+1 & 0 & 0 \end{bmatrix}$$

1) $a=-1$: $\operatorname{rang}(A_a+2I) = 2 \Rightarrow J(A_a)$ ima dve kletki

$$(A_{-1}+2I)^2 = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 1 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & -1 \end{bmatrix}^2 = \begin{bmatrix} 1 & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \Rightarrow \operatorname{rang}(A_{-1}+2I) = 1 \Rightarrow m_{A_{-1}}(\lambda) = (\lambda+2)^3$$

iščemo $v_1^{(3)} \in \ker(A_{-1}+2I)^3 \setminus \ker(A_{-1}+2I)^2$, npr. $v_1^{(3)} = (1,0,0,0)^T \Rightarrow v_1^{(2)} = (A_{-1}+2I)(1,0,0,0)^T = (0,1,0,-1)^T$

$v_1^{(1)} = (A_{-1}+2I)(0,1,0,-1) = (1,-1,0,0)^T$

iščemo lastni vektor za A_{-1}, ki ni $v_1^{(1)}$, dobimo ga iz sistema $A_{-1}+2I \sim \begin{bmatrix} 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$ npr. $v_2 = (0,0,1,1)$

$$J(A_{-1}) = \begin{bmatrix} -2 & 1 & & \\ & -2 & 1 & \\ & & -2 & \\ & & & -2 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

2) $a \neq -1$: $\operatorname{rang}(A_a+2I) = 3 \Rightarrow J(A_a)$ ima eno kletko

$$(A_a+2I)^2 = \begin{bmatrix} 1 & 1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ a+1 & a+1 & -2a-2 & 2a+2 \\ a+1 & a+1 & -2a-2 & 2a+2 \end{bmatrix} \qquad (A_a+2I)^3 = \begin{bmatrix} 0 & & & \\ 0 & & & \\ -a-1 & & & \\ & & & \end{bmatrix} \Rightarrow v^{(4)} = (1,0,0,0)^T$$

$v^{(3)} = (A_a+2I)(1,0,0,0) = (0,1,0,-1)^T$

$v^{(2)} = (A_a+2I)(0,1,0,-1)^T = (1,-1,a+1,a+1)$

$v^{(1)} = (A_a+2I)(1,-1,a+1,a+1) = (0,0,-a-1,-a-1)$

$$J(A_a) = \begin{bmatrix} -2 & 1 & & \\ & -2 & 1 & \\ & & -2 & 1 \\ & & & -2 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ -a-1 & a+1 & 0 & 0 \\ -a-1 & a+1 & -1 & 0 \end{bmatrix}$$