(11) DOKAŽI BERNOULLIJEVO NEENAKOST:

ZA VSAK $n \in \mathbb{N}$ IN $a > -1$ VELJA

$$(1+a)^n \geq 1+na.$$

BAZA INDUKCIJE $n=1$:

$$(1+a)^1 \geq 1+a \quad \checkmark$$

INDUKCIJSKI KORAK $n \to n+1$:

$$(1+a)^{n+1} = (1+a)^n \cdot (1+a) \overset{\text{I.P.}}{\geq} (1+na)(1+a)$$

ŽELIMO TOREJ DOKAZATI, DA VELJA:

$$(1+na)(1+a) \geq 1+(n+1)a$$

$$1+na+a+na^2 \geq 1+na+a$$

$$na^2 \geq 0$$

TO VEDNO DRŽI.

VPRAŠANJE:

KJE SMO UPORABILI DEJSTVO, DA JE $a > -1$?

ODGOVOR:

TAKRAT, KO SMO UPORABILI I. P.

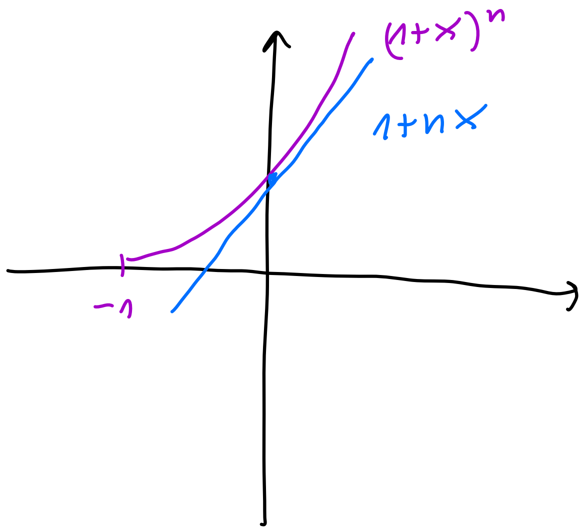
RES, ČE BI BIL $a < -1$, BI VELJALO

$$(1+a)^n \geq 1+na$$

$$(1+a)^{n+1}(1+a) \leq (1+na)(1+a)$$

ZNAK BI SE OBRNIL.

GEOMETRIJSKI POMEN:



PREMICA $1+nx$ JE TANGENTA NA $(1+x)^n$ IN VEDNO POD KRIVULJO.

(12) DOKAŽI, DA ZA VSE $n \in \mathbb{N}$ VELJA:

$$n! \leq 2 \cdot \left(\frac{n}{2}\right)^n$$

BAZA $n=1$:

$$1! \leq 2 \cdot \left(\frac{1}{2}\right)^1 \quad \checkmark$$

IND. KORAK $n \to n+1$:

$$(n+1)! = (n+1)\, n! \overset{\text{I.P.}}{\leq} (n+1) \cdot 2 \cdot \left(\frac{n}{2}\right)^n$$

DOVOLJ BO POKAZATI, DA JE

$$(n+1) \cdot 2 \cdot \left(\frac{n}{2}\right)^n \leq 2 \cdot \left(\frac{n+1}{2}\right)^{n+1}$$

PO KRAJŠANJU ČLENOV DOBIMO:

$$n^n \leq \frac{1}{2} \cdot (n+1)^n \quad / \cdot \frac{2}{n^n}$$

$$2 \leq \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$$

$$2 \leq \underbrace{1+\binom{n}{1} \cdot \frac{1}{n}}_{=2} + \underbrace{\binom{n}{2} \cdot \frac{1}{n^2} + \ldots + \frac{1}{n^n}}_{\text{POZITIVNI ČLENI}} \quad \checkmark$$

13 DOKAŽI:

(a) $10 \mid 2^{2^n} - 6$ ZA $n \geq 2$.

(b) $7 \mid 13^{2n} + 6$

(a) BAZA $n=2$:

$$2^{2^2} - 6 = 2^4 - 6 = 16 - 6 = 10$$

DELJIVO Z 10.

KORAK $n \to n+1$:

INDUKCIJSKA PREDPOSTAVKA JE $10 \mid 2^{2^n} - 6$, KAR LAHKO ZAPIŠEMO KOT $2^{2^n} - 6 = 10k$ ZA NEKO ŠTEVILO $k \in \mathbb{N}$. TO UPORABIMO V IZRAZU ZA $n+1$:

$$2^{2^{n+1}} - 6 = 2^{2^n \cdot 2} - 6 = \left(2^{2^n}\right)^2 - 6 =$$

$$\overset{\text{I.P.}}{=} (10k+6)^2 - 6 = 100k^2 + 120k + 36 - 6 =$$

$$= 100k^2 + 120k + 30 = 10 \cdot (10k^2 + 12k + 3)$$

TOREJ JE TUDI $2^{2^{n+1}}$ VEČKRATNIK ŠTEVILA 10.

(b) BAZA $n=1$:

$$13^2 + 6 = 169 + 6 = 175 = 7 \cdot 25 \ ✓$$

KORAK $n \to n+1$:

UPORABIMO ISTI TRIK KOT ZGORAJ

$$13^{2n} + 6 = 7k$$

$$13^{2(n+1)} + 6 = 13^{2n+2} + 6 = 13^2 \cdot 13^{2n} + 6 =$$

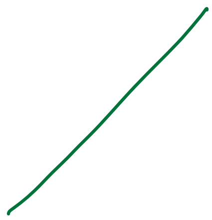
$$= 169 \cdot (7k - 6) + 6 = 169 \cdot 7k - 169 \cdot 6 + 6 =$$

$$= 169 \cdot 7k - 168 \cdot 6 = 7 \cdot (169k - 24 \cdot 6) \ ✓$$

(14) DOKAŽI, DA n PAROMA NEVZPOREDNIH PREMIC, OD KATERIH SE NOBENE TRI NE SEKAJO V ISTI TOČKI, RAVNINO RAZDELI NA $\frac{n(n+1)}{2}+1$ DELOV.

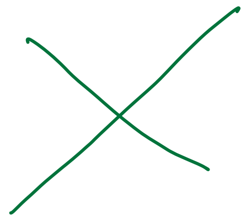
R: OGLEJMO SI PRVA TRI NARAVNA ŠTEVILA.

$n=1$:



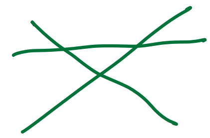
$\frac{1\cdot 2}{2}+1=2$ DELA ✓

$n=2$:



$\frac{2\cdot 3}{2}+1=4$ DELI ✓

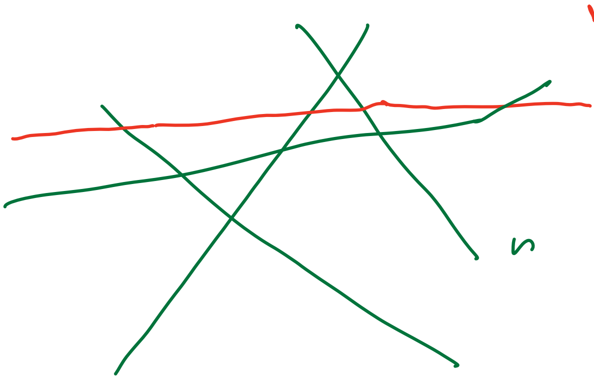
$n=3$



$\frac{3\cdot 4}{2}+1=7$ DELOV ✓

KORAK $n \to n+1$

DODATNA PREMICA



n PREMIC PO PREDPOSTAVKI RAVNINO DELI NA $\frac{n(n+1)}{2}+1$ DELOV. ENA DODATNA PREMICA PA PRINESE $n+1$ DODATNIH DELOV.

VSOTA: $\frac{n(n+1)}{2}+1+(n+1)=\frac{n(n+1)+2(n+1)}{2}+1=$

$=\frac{(n+1)(n+2)}{2}+1$ DELOV ✓

(15) DOKAŽI BINOMSKO FORMULO: ZA $a,b \in \mathbb{R}$ IN $n \in \mathbb{N}$ VELJA

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \dots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

R: BAZA $n=1$:

$$(a+b)^1 = \binom{1}{0}a + \binom{1}{1}b = a+b \quad \checkmark$$

IND. KORAK $n \to n+1$:

$$(a+b)^{n+1} = (a+b)(a+b)^n = (a+b)\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k =$$

$$= \sum_{k=0}^{n}\binom{n}{k}a^{n+1-k}b^k + \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k+1} =$$

$$= \binom{n}{0}a^{n+1} + \binom{n}{1}a^n b + \dots + \binom{n}{n}ab^n + \binom{n}{0}a^n b + \binom{n}{1}a^{n-1}b^2 + \dots + \binom{n}{n}b^{n+1} =$$

$$= a^{n+1} + \left(\binom{n}{1}+\binom{n}{0}\right)a^n b + \left(\binom{n}{2}+\binom{n}{1}\right)a^{n-1}b^2 + \dots + b^{n+1}$$

$$= a^{n+1} + \binom{n+1}{1}a^n b + \binom{n+1}{2} + a^{n-1}b^2 + \dots + b^{n+1}$$

TO SI ŽELIMO DOBITI NA KONCU.
DA BO TO RES, MORAMO DOKAZATI

$$\binom{n}{\ell} + \binom{n}{\ell-1} = \binom{n+1}{\ell} \quad \text{ZA} \quad \ell \in \{1, 2, \dots, n\}$$

DOKAZ:

$$\binom{n}{\ell} + \binom{n}{\ell-1} = \frac{n!}{(n-\ell)!\,\ell!} + \frac{n!}{(n-\ell+1)!\,(\ell-1)!} =$$

$$= \frac{n!\cdot(n-\ell+1) + n!\,\ell}{(n-\ell+1)!\,\ell!} = \frac{n!\,(n-\cancel{\ell}+1+\cancel{\ell})}{(n-\ell+1)!\,\ell!} =$$

$$= \frac{(n+1)!}{((n+1)-\ell)!\,\ell!} = \binom{n+1}{\ell}$$

TO SMO ŽELELI DOBITI!