# Real Numbers, Ratios and Binary

1. (Mixing paint) A shade of green paint is produced using 4 litres of blue paint and 2 litres of yellow paint. How much blue paint is needed to mix with 8 litres of yellow paint?

2. Prices are not always directly proportional to the number of items. Which of the following are in direct proportion?

    1. Emulsion paint at £7.99 for 2.5 litres and £14.99 for 5 litres.
    2. 5 packs of ceramic wall tiles for £50 and 3 packs for £30.
    3. 1 loaf of bread for 75p and 4 loaves for £3.
    4. 50 g of chocolate for 40p and 200 g of chocolate for £1.20.

3. A-series sheets of paper have been designed in such a way that when you cut them in half, you get two perfectly identical pieces of the next biggest size. For example, if you cut an A3 sheet in half, you will get two A4-sized sheets. Determine the ratio between their length and breadth.

4. Read the following theorem and its proof.

    **Theorem 1.** $\sqrt{20}$ *is irrational.*

    *Proof.* Suppose there are integers $m$ and $n$ such that $\sqrt{20} = m/n$. Without loss of generality we may assume that $m$ and $n$ have no factors in common. Now $m^2 = 20n^2$. Hence 5 is a factor of $m^2$, and so 5 must be a factor of $m$. We can therefore write $m = 5k$, for some integer $k$. Then $25k^2 = m^2 = 20n^2$, or $5k^2 = 4n^2$. Hence 5 is a factor of $n^2$, and hence of $n$. But then 5 is a factor of $m$ and $n$, contradicting our assumption. $\square$

    Now answer the following questions.
    (a) What method of proof is used here?
       (A) Proof by transfinite induction,
       (B) Proof by contradiction,
       (C) Proof by example,
       (D) Proof by induction.

    (b) How is $\sqrt{20}$ defined?

(A) The smallest real number $x$ such that $x^2 = 20$,

(B) Any real number $x$ such that $x^2 = 20$,

(C) The unique positive real number such $x^2 = 20$,

(D) The unique positive rational number such that $x^2 = 20$.

(c) When is a real number irrational?

(A) When it can be written as the quotient of two integers,

(B) When it cannot be written as the quotient of two integers,

(C) When it is not the zero of a polynomial with integer coefficients,

(D) When it can be written as $\sqrt{n}$ for some natural number $n$.

(d) Why may we assume that $m$ and $n$ have no factors in common?

(A) Because this holds when $m/n$ is in reduced form,

(B) Because $m/n$ is assumed to be a square root,

(C) The denominator and numerator of a fraction always satisfy this property,

(D) Because otherwise the proof does not work.

(e) Given that 5 is a factor of $m^2$ how does it follow that 5 is a factor of $m$?

(A) Because there are infinitely many prime numbers,

(B) Because every factor of $m^2$ is a factor of $m$,

(C) Because the square root of 25 is divisible by 5,

(D) Because every natural number can be written uniquely as a product of primes.

(f) Why does $5k^2 = 4n^2$ imply that 5 is a factor of $n^2$?

(A) Because 4 is a factor of $k$,

(B) Because $4/5 = 0.8$,

(C) Because every natural number can be written uniquely as a product of primes,

(D) Because there are infinitely many primes.

(g) Which assumption is directly contradicted?

(A) That $m$ and $n$ have no factors in common,

(B) The assumption that $\sqrt{20} = m/n$,

(C) The assumption that 5 is a factor of $m^2$,

(D) The assumption that 5 is a factor of $m$.

(h) The equation $m^2 = 20n^2$ also implies that 2 is a factor of $m^2$. Does the same proof work if 2 is used as the chosen factor of $m^2$?

(A) Yes,

(B) No.

5. (Proportion solving) Decide whether the following situations show values that increase in proportion or not. The hints suggest doubling one amount and deciding whether the other amount doubles as well. If the amounts are in proportion then when one amount doubles the other amount will also double.

   a) The cost of a monthly mobile phone contract and the number of minutes called. Hint: if you double the number of minutes, would you pay double the amount?

   b) The amount of income tax paid and salaries. Hint: does a person who earns twice as much as another person pay twice as much tax?

   c) The amounts of blue and yellow paint to make a particular shade of green. Hint: if you double the amount of blue paint, would you double the amount of yellow paint?

   d) The population of a country and the land area. Hint: does a country with double the amount of people as another have twice the land area?

6.   a) Share a prize of £50 between in the ratio 2:3.

     b) Share £75 in the ratio 2:3.

     c) Share £60 in the ratio 4:5:1.

7. Two car dealers have started a business. They expect it to be a success but have agreed to share all profits and losses equally, in the same ratio as they've invested their money to buy stock and premises. Dealer A has invested £180,000 and Dealer B has invested £120,000. How would they share:

   a) a profit of £70,000

   b) a loss of £8,000

A third dealer (Dealer C) had joined the partnership with the same agreement about sharing profits and losses. Dealer C had invested £300,000, how would they have shared the profits and losses above?

8. Convert each of the following binary numbers to decimal.

   a) 100                          d) 10001000

   b) 1111                         e) 1101110

   c) 1101010                      f) 11111111

9. Convert each of the following decimal numbers to binary.

   a) 21                           c) 113

   b) 36                           d) 82

10. Tom is trapped on the top floor of a department store. It's just before Christmas and he wants to get home with his presents. What can he do? He has tried calling, even yelling, but there is no one around. Across the street he can see some computer person still working away late into the night. How could he attract her attention? Tom looks around to see what he could use. Then he has a brilliant idea - he can use the Christmas tree lights to send her a message! He finds all the lights and plugs them in so he can turn them on and off. He uses a simple binary code, which he knows the woman across the street is sure to understand. Can you work it out?