Jedra, kvarki in leptoni (Nuclei, quarks, and leptons)

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Work in progress!

1 Lie Groups and its Representations

Lie groups are continuous groups, parametrized by real parameters α_k:

$$g(\alpha) = e^{i\alpha_k T_k}, \tag{1}$$

such that $g(0) = e$. T_k are Hermitian generators of the group. The group product is defined as

$$\begin{aligned} g(\beta)g(\alpha) &= e^{i\beta\cdot T} e^{i\alpha\cdot T} \\ &= e^{i(\beta+\alpha)\cdot T - \frac{1}{2}[\beta\cdot T, \alpha\cdot T] + \cdots} \end{aligned} \tag{2}$$

where we have used the Baker-Campbell-Hausdorff formula,

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]-\frac{1}{12}[B,[A,B]]+\cdots}. \tag{3}$$

The product (2) should be member of the group and thus the commutators of generators have to be proportional to generators:

$$[\beta\cdot T, \alpha\cdot T] = i\gamma(\beta,\alpha)\cdot T. \tag{4}$$

Parameters γ depend on arbitrary α in β. Also all further terms of the exponent (2) are nested commutators thus the condition (4) is sufficient to show that the group is closed under multiplication. If we choose for α and β unity vectors in directions a and b,

$$\begin{aligned} \beta^{(a)} &= (0, \ldots, \overset{a}{1}, \ldots, 0) \\ \alpha^{(b)} &= (0, \ldots, \underset{b}{1}, \ldots, 0), \end{aligned} \tag{5}$$

we obtain a commutation relations (*Lie algebra*)

$$[T_a, T_b] = i\gamma_c^{(ab)} T_c. \tag{6}$$

Here $f_{abc} \equiv \gamma_c^{(ab)}$ are called the structure constants of the group that unambiguously define the structure of the Lie group. Working in a specific representation (of a compact Lie group), we can always choose generators to be orthogonal in the following sense:

$$\mathrm{Tr}(T_a T_b) = \frac{1}{2}\delta_{ab}. \tag{7}$$

A unitary representation of the group is a set of unitary matrices $U(g(\alpha))$ that faithfully reproduces the group multiplication law:

$$U(g(\alpha))U(g(\beta)) = U(g(\alpha)g(\beta)). \tag{8}$$

A shorthand notation is introduced:

$$U(\alpha) \equiv U(g(\alpha)) = 1 + i\alpha \cdot T + \cdots , \tag{9}$$

where T_a are now Hermitian matrices for a given representation.

Exercise 1.1

Show that the inverse element is $g(\alpha)^{-1} = g(-\alpha)$.

1. Prove that the structure constants are antisymmetric in the first two indices, $f_{abc} = -f_{bac}$ and that they are real.

2. Show that we can calculate the structure constants directly from the generators as

$$f_{abc} = -2i\mathrm{Tr}\left([T_a, T_b]T_c\right).$$

   Use the above relation to show complete antisymmetry of f_{abc}.

3. Check the Jacobi identity

$$[T_a, [T_b, T_c]] + [T_b, [T_c, T_a]] + [T_c, [T_a, T_b]] = 0.$$

2 Rotation group $SO(N)$

The defining representation of the rotation group $SO(N)$ is defined as:

$$SO(N) = \{O \in O(N) \quad | \quad \det O = 1\}, \tag{10}$$

$$O(N) = \left\{O \in GL_N(\mathbb{R}) \quad | \quad O^T O = 1\right\}, \tag{11}$$

The above groups are also the defining representations of dimension N.

2.1 Eigenstates and representations of $SO(3)$ (problems)

Problems:

1. Show that if O is an orthogonal matrix, $O \in O(N)$, then $\det O = \pm 1$.

2. Show that the scalar product $v^T w$ of two vectors $v, w \in \mathbb{R}^N$ is not changed upon rotating both vectors by O.

3. Determine the number of real parameters needed to specify orthogonal matrix O. *Sol.:* $N(N-1)/2$

4. Assume O is an orthogonal matrix sufficiently close to identity, $O = 1 + A$, where A is infinitesimal. Show that A is antisymmetric matrix. What is the dimension of the vector space of $N \times N$ antisymmetric matrices?

5. For rotations in the plane, i.e. $SO(2)$, write explicit form of small rotation, $O = 1 + \epsilon A$, where ϵ is infinitesimal. Determine the generator A. Is the group Abelian or non-Abelian? Derive the explicit matrix for rotation by finite angle α.

6. Find the generator matrices of $SO(3)$ for the $j = 1$ representation. Use them to find the explicit form of the finite rotation matrix $d^1_{m'm}(\theta) = \langle jm' \,|\, e^{-i\theta J_2} \,|\, jm \rangle$.

7. Consider how a vector field $\boldsymbol{A}(\boldsymbol{r}) = (A_1(\boldsymbol{r}), A_2(\boldsymbol{r}), A_3(\boldsymbol{r}))$ transforms under $SO(3)$ rotation R, that acts on vectors as $\boldsymbol{r} \to \boldsymbol{r}' = R\boldsymbol{r}$. The rotated field is then $\boldsymbol{A}'(\boldsymbol{r}) = R\boldsymbol{A}(R^{-1}\boldsymbol{r})$ and you can consider R that corresponds to an infinitesimal rotation around the axis $d\boldsymbol{\phi}$. Show that in this case the rotated field can be written as
$$\boldsymbol{A}'(\boldsymbol{r}) = U(d\boldsymbol{\phi})\boldsymbol{A}(\boldsymbol{r}) = [1 - i\, d\boldsymbol{\phi} \cdot (\boldsymbol{\ell} + \boldsymbol{s})]\, \boldsymbol{A}(\boldsymbol{r}).$$
Determine operators $\boldsymbol{\ell}$ and $\boldsymbol{s}$. Show that spin of a vector field $\boldsymbol{A}$ is $s = 1$ and that matrices $\boldsymbol{s}$ are equivalent to the usual spin-1 generators with diagonal J_3 (show that the matrices are similar, i.e. related by basis transformation).

8. Explicitly write out direct product of three $s = 1/2$ quarks, i.e., spin wave-function of baryons.

3 $SU(2)$ group and isospin

The $SU(2)$ group in the defining representation can be written as an exponent of the traceless Hermitian matrix:
$$U(\boldsymbol{\alpha}) = \exp(-i\alpha_a \sigma_a/2), \qquad a = 1, 2, 3. \tag{12}$$
The $SU(2)$ algebra is identical to the $SO(3)$ algebra with structure constants $f_{abc} = \epsilon_{abc}$.

3.1 Problems

1. Show that the group $SU(2)$ has the same algebra (structure constants) as $SO(3)$. Thus same state labels and ladder operators can be used on eigenstates of isospin.

2. Write down the isospin rotation that transforms between nucleon doublet eigenstates, $n \to p$, and the other rotation that transforms $p \to n$.

3. Deuteron is a bound state of a proton and a neutron. We do not observe bound states of pp or nn. What are the possible states of spin and orbital angular momentum if the total wave function should obey the Pauli exclusion principle under interchange of nucleons? Narrow down the selection if you take into account that deuteron has positive parity and that its spin is $j = 1$.

4. Calculate the ratio of deuteron production cross-section rates
$$\sigma(pn \to d\pi) : \sigma(pp \to d\pi) : \sigma(nn \to d\pi).$$

4 $SU(3)$ group role in flavour of light quarks and chromodynamics

1. Count the number of generators T^a of $SU(3)$ and write down the Gell-Mann matrices $\lambda^a = 2T^a$.

2. Check that there are three $SU(2)$ subalgebras of $SU(3)$: (T_1, T_2, T_3), $(T_4, T_5, (3/4)Y - (1/2)T_3) \equiv (V_1, V_2, V_3)$, $(T_6, T_7, (3/4)Y + (1/2)T_3) \equiv (U_1, U_2, U_3)$.

3. (m, n) notation for irreducible representation of $SU(3)$ denotes Check the dimension formula $\dim = (1/2)(m+1)(n+1)(m+n+2)$ on the case of baryon octet and decuplet. Compare the result with the Young tableau's "factors over hooks".

4. Use Young tableaux to determine direct product of baryon states $3 \otimes 3 \otimes 3$

$$\square \otimes \square \otimes \square$$

5. Direct products of $SU(3)$ representations. The fundamental representation ψ^i, $i = 1, 2, 3$ (or $i = u, d, s$, or $i = R, G, B$) or simply 3, transforms under the $U \in SU(3)$ as

$$\psi^i \mapsto \psi'^i = U^i_{\ j} \psi^j, \quad (\psi \mapsto U\psi).$$

A single upper index (fundamental) is denoted by a single box Young tableau, $\square$. On the other hand, antifundamental representation $\bar{3}$ is defined as the transformation of complex conjugate ψ^*:

$$\psi^{*i} \mapsto \psi^{*'i} = (U^i_{\ j})^* \psi^{*j}, \quad (\psi^* \mapsto U^* \psi^*, \text{ or } \psi^\dagger \mapsto \psi^\dagger U^\dagger).$$

By convention we label the antifundamental reprentation with lower indices:

$$\phi_i \mapsto \phi'_i = U_i^{\ j} \phi_j,$$

where $U_i^{\ j} \equiv (U^i_{\ j})^*$.

(a) Use the unitarity of U to calculate $U_k^{\ j} U^i_{\ j}$ (similarly for $U_j^{\ k} U^j_{\ i}$). Show then that $\phi_i \psi^i$ is invariant. This is equivalent to saying that tensor δ^i_k is invariant.

(b) Another invariant tensor is antisymmetric tensor ϵ_{ijk} with $\epsilon_{123} = 1$. Use the property $\det U = 1$ (unimodularity) to show invariance of ϵ under $SU(3)$ transformation:

$$\epsilon'_{ijk} = U_i^{\ m} U_j^{\ n} U_k^{\ o} \epsilon_{mno}$$

(Hint: Check that the right-hand side is completely antisymmetric in indices (ijk), thus it must be proportional to ϵ_{ijk}.) Equivalent statement is that antisymmetric "contraction" of three fundamentals $\square$, $\epsilon_{ijk}\psi^i \phi^j \omega^k$ is a singlet, or in terms of Young tableau:

$$1 = \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array}.$$

(c) We can lower the indices. Show that antisymmetric "contraction" of two upper indices results in an object $\chi_k = \epsilon_{kij}\psi^i \phi^j$ that indeed transforms as antifundamental. Or in terms of Young tableau:

$$\bar{3} = \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}.$$

Employ unitarity and unimodularity of U.

6. Write down the wavefunction of the proton, using the fact that total spin and isospin should be 1/2 and that the total wavefunction of flavour and spin should be completely symmetric under interchange of any pair of quarks.

7. Try to write down completely antisymmetric wave function of the proton. In this case color part of the wavefunction would not be required. It turns out this possibility is in conflict with the experimentally determined magnetic moment of the proton, $\mu_p^{\mathrm{exp}} = 2.79\mu_N$ (See also Exercise 2.18 in [1]).

8. Determine the wavefunction of Σ^0 and Λ baryons. Λ is an isospin singlet, Σ^0 can be obtained from Σ^+. Check that the magnetic moment of the Λ baryon equals the magnetic moment of the s-quark: $\mu_s = -e_0\hbar/(6m_s) = -\mu_N\, m_p/(3m_s)$.

9. Calculate the probability of finding a d quark with spin $\uparrow$ in baryonic state $|\Xi, \downarrow\rangle$. *(Answer:* 1/3*)*.

10. We study strong decays $\Sigma^{*0} \to \Sigma^+\pi^-$ and $\Xi^{*0} \to \Xi^-\pi^+$. Quantum numbers of the states are given in the following table:

	B	I	I_3	Y	U	U_3
Σ^{*0}	1					
Ξ^{*0}	1					
Σ^+	1					
Ξ^-	1					
$\pi^\pm$	0					

    (a) Fill out missing values in the table.

    (b) Write down flavour, spin, and colour part of the wavefunctions of Σ^{*0} and Ξ^{*0}, both members of the baryon decuplet with $J^P = (3/2)^+$.

    (c) Calculate the ratio of decay widths $\Gamma(\Sigma^{*0} \to \Sigma^+\pi^-)/\Gamma(\Xi^{*0} \to \Xi^-\pi^+)$. Employ the invariance of strong interactions under U-spin.

11. Hyperfine splitting between baryon octet and decuplet. Calculate parameter κ for a hyperfine splitting model, describing a baryon masses:

$$M_{\mathrm{barion}} = m_1 + m_2 + m_3 + \kappa\left(\frac{\vec{s}_1\cdot\vec{s}_2}{m_1m_2} + \frac{\vec{s}_2\cdot\vec{s}_3}{m_2m_3} + \frac{\vec{s}_3\cdot\vec{s}_1}{m_3m_1}\right),$$

    where take into account $m_q = m_u = m_d = 300$ MeV and $m_s = 500$ MeV. Measured mass splitting between $J^P = (3/2)^+$ and $(1/2)^+$ states (both isotriplets) is $\Delta M = M_{\Sigma^*} - M_\Sigma = 195$ MeV.

12. Assume there exist baryons composed of 4 quarks. Their wave function is

$$|\psi\rangle_{\text{flavour, spin}} \otimes |\chi\rangle_{\text{color}},$$

    where $|\psi\rangle_{\text{flavour, spin}}$ is symmetric to quark interchange.

(a) Determine dimension of baryonic multiplet with spin 2 for flavour symmetry $SU(3)_{\text{flavour}}$ of quarks $\{u, d, s\}$. Write down wavefunction of state $|\psi\rangle_{\text{flavour, spin}}$ with maximal Y and T_3, and sketch all states in plane (T_3, Y). Take into account that the baryon number of a quark is $1/4$. Label quark composition of each state.

(b) Determine wave function of state $|\psi\rangle_{\text{flavour, spin}}$ with quantum numbers $(T_3, Y) = (1/2, 0)$ with spin projection $S_3 = 1$. Determine the probability that both u-quarks have spin $\uparrow$.

(c) Would such baryon states be possible with $SU(3)$ group of strong (colour) interactions? Note that $|\chi\rangle_{\text{colour}}$ should be completely antisymmetric.

13. Write down complete wavefunctions of the lightest pseudoscalar mesons $\bar{3} \otimes 3 = 8 \oplus 1$. Argue, why these states with $\ell = 0$ have $J^P = 0^-$. Determine the charge conjugation parity of neutral mesons, like η_8 and π^0.

14. Write down wavefunctions of the vector meson octet. Determine the magnetic moment of $\rho^\pm$ and ρ^0. *(Answer:* $\pm(\mu_u - \mu_d), 0$*.)*

15. η and η', wave functions, mixing angle.

16. Argue with color factors due to single gluon exchanges that the nonrelativistic potential between $q\bar{q}$ in the color singlet state is attractive. Alternatively, suppose that mesons were in the color octet configuration. In this case, show that the potential is repulsive for a meson with color structure $1/\sqrt{2}(R\bar{R} - G\bar{G})$.

5 Lorentzova in Poincaré-jeva grupa

Posplošitev Galilejeve transformacije na Lorentzovo omogoča, da se pri transformacijah med inercialnimi sistemi ohranja hitrost svetlobe, $c = 1$. Dogodke x^μ opišemo s četverci v prostoru Minkowskega

$$x^\mu = (t, \boldsymbol{x}) = (x^0, x^1, x^2, x^3). \tag{13}$$

Lorentzova transformacija štirivektorja je

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu. \tag{14}$$

Za potovanje elektromagnetnega valovanja velja $|\boldsymbol{x}| = t$ ali $t^2 - \boldsymbol{x}^2 = 0$, kar lahko zapišemo s produktom $x \cdot x = 0$, kjer je skalarni produkt med dvema četvercema definiran kot $x \cdot y = x^\mu \eta_{\mu\nu} y^\nu$. Vpeljali smo metrični tenzor

$$\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)_{\mu\nu} \tag{15}$$

s kovariantnima (spodnjima) indeksoma. Množenje s kovariantnim metričnim tenzorjem naredi iz kontravariantnega četverca x^μ kovarianten četverec $x_\mu \equiv \eta_{\mu\nu} x^\nu = (t, -\boldsymbol{x})$. *Določi komponente* $\eta^{\mu\nu}$ *ter pokaži, da je* $\eta^\mu{}_\nu = \delta^\mu_\nu$. *Izračunaj* $\eta^{\mu\nu}\eta_{\mu\nu}$.

Princip posebne relativnosti potem prevedemo na zahtevo, da je skalarni produkt invarianten na Lorentzove transformacije, torej $x' \cdot y' = x \cdot y$:

$$\begin{aligned} \Lambda^\mu{}_\rho x^\rho \Lambda^\nu{}_\omega y^\omega \eta_{\mu\nu} &= x^\rho y^\omega \eta_{\rho\omega}, \\ \Rightarrow \Lambda^\mu{}_\rho \Lambda^\nu{}_\omega \eta_{\mu\nu} &= \eta_{\rho\omega}. \end{aligned} \tag{16}$$

Iz zahteve (16) *določi komponente inverzne Lorentzove transformacije* $(\Lambda^{-1})^\mu{}_\nu$. Matrike Λ definirajo grupo $SO(1,3)$.

5.1 Problems

1. Write down the infinitesimal Lorentz transformation $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}$. How many independent $\omega_{\mu\nu}$ are there? Write down the generators $J^{\mu\nu}$ of Lorentz transformations $\Lambda^{\alpha}{}_{\beta} = \left[\exp(\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu})\right]^{\alpha}{}_{\beta}$ (notice how matrices $[J^{\mu\nu}]^{\alpha\beta}$ should be antisymmetric and purely imaginary, therefore Hermitian). We identify rotation generators with $J^i = \frac{1}{2}\epsilon^{ijk}J^{jk}$ and boosts with $K^i = J^{0i}$.

2. Starting from the infinitesimal form of boost along 1-direction, derive the explicit matrix form of finite boost, i.e., use $\omega_{01} = -\omega_{10} = \omega$ and use expand the exponential. What is the correspondence of ω with β and γ?

3. Show that Lorentz transformations ($SO(1,3)$) can be implemented by $SL(2,\mathbb{C})$, acting on Hermitian matrix $V = v^{\mu}\sigma_{\mu}$ (do the parameter counting!). Here $\sigma^{\mu} = (1, \boldsymbol{\sigma})$, $\bar{\sigma}^{\mu} = (1, -\boldsymbol{\sigma})$. Determine the inverse transformation, from $V \to v^{\mu}$. Note that v^2 does not change by $SL(2,\mathbb{C})$ transformations A acting as $AVA^{\dagger}$. For given A determine the Lorentz transformation Λ.

4. Demonstrate that the 4-dimensional representation $J^{\mu\nu} = \frac{i}{4}[\gamma^{\mu}, \gamma^{\nu}]$, where the anticommutator of gamma matrices is $\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu}$, satisfies the Lorentz algebra:

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(\eta^{\nu\rho}J^{\mu\sigma} + \eta^{\mu\sigma}J^{\nu\rho} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho}\right). \tag{17}$$

5. Transformation of the Dirac bispinor $\psi(x) \mapsto \psi'(x') = U(\omega)\psi(x)$ in the Dirac equation rotates the gamma matrices as $\gamma^{\nu} \mapsto U\Lambda^{\nu}{}_{\mu}\gamma^{\mu}U^{-1}$. Using the infinitesimal ω show that γ^{ν} is not changed under this transformation. Show the equivalent relation, $U^{-1}\gamma^{\nu}U = \Lambda^{\nu}{}_{\mu}\gamma^{\mu}$.

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi(x) = 0, \tag{18}$$

   is indeed an invariant equation.

6. Determine how $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^{\mu}, \gamma^{\nu}]$ transforms under Hermitian conjugation.

7. Show that $\bar{psi}\sigma^{\mu\nu}\psi$ transforms under Lorentz transformation as a two-index Lorentz tensor.

8. Use the parity operator $U_P = \gamma^0$ to show that $\bar{\psi}\gamma^5\psi$ is a pseudoscalar and that $\bar{\psi}\gamma^{\mu}\gamma^5\psi$ is a axial-vector.

9. Calculate the following traces:

$$\mathrm{Tr}(\gamma^{\mu}\gamma^{\nu}\gamma^{\alpha}\gamma^{\beta}) =$$
$$\mathrm{Tr}(\gamma^{\mu}\gamma^{\nu}\gamma^{\alpha}\gamma^{\beta}\gamma^5) =$$

5.2 Poincaré-jeve transformacije

Splošne transformacije iz danega inercialnega sistema v katerikoli drug inercialni sistem tvorijo Poincaré-jevo grupo. Dogodki se transformirajo kot

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu} + a^{\mu} \tag{19}$$

kjer je Λ Lorentzova transformacija in a konstanten vektor. Kjer ne bo nujno bomo zamolčali indekse in pisali $x' = \Lambda x + b$. Sledeč poglavje 2 [2] je komponiranje Poincaré-jev takšno:

$$x'' = \Lambda' x' + b = \Lambda' \Lambda x + \Lambda' a + b, \tag{20}$$

torej

$$\begin{aligned} (\Lambda', b) \circ (\Lambda, a) &= (\Lambda' \Lambda, \Lambda' a + b), \\ \Rightarrow (\Lambda, a)^{-1} &= (\Lambda^{-1}, -\Lambda^{-1} a) \end{aligned} \tag{21}$$

5.3 Problems

1. Show that $P^2 = P_\mu P^\mu$ commutes with the generators of the Poincaré group. Employ the commutation relations of the Poincaré algebra (see Ch. 2.4 in [2]):

$$\begin{aligned} [J^{\mu\nu}, J^{\rho\sigma}] &= i\left(\eta^{\nu\rho} J^{\mu\sigma} + \eta^{\mu\sigma} J^{\nu\rho} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\sigma} J^{\mu\rho}\right), \\ [P^\mu, J^{\rho\sigma}] &= i\left(\eta^{\mu\rho} P^\sigma - \eta^{\mu\sigma} P^\rho\right), \\ [P^\mu, P^\rho] &= 0. \end{aligned}$$

   Calculate the action of P^2 on the eigenstate $|\boldsymbol{p}, j, j_z\rangle$.

2. Derive the matrix form of the momentum generator, $(P_\mu)^\alpha{}_\beta$, acting on four-vectors. Compare infinitesimal transformation $x'^\alpha = x^\alpha + a^\alpha$ with $[\exp(i a_\mu P^\mu)]^\alpha{}_\beta x^\beta$. *(Sol.: $(P_\mu)^\alpha{}_\beta = -i\partial_\mu \delta^\alpha_\beta$)*

3. Pauli-Lubanski vector is defined as

$$W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^\sigma,$$

   where $\epsilon_{0123} = -1$. Calculate the action of W_μ and $W^2 = W_\mu W^\mu$ on the one-particle state $|\boldsymbol{p}, j, j_z\rangle$.

6 EM scattering of electrons on μ^-, p

6.1 $e^-\mu^- \to e^-\mu^-$

6.2 Problems

1. Calculate the invariant amplitude $\mathcal{M}$ for elastic scattering $e(k)\mu(p) \to e(k')\mu(p')$. The muon is initially at rest, whereas electron can be considered ultrarelativistic. Do the spin sums and averaging to find $\overline{|\mathcal{M}|^2}$.

2. Derive the two-body phase space element for the above case.

1

Literatura

[1] F. Halzen and A. D. Martin. *Quarks and Leptons: An Introductory Course in Modern Particle Physics*. 1984.

[2] S. Weinberg. *The Quantum theory of fields. Vol. 1: Foundations*. Cambridge University Press, 6 2005.