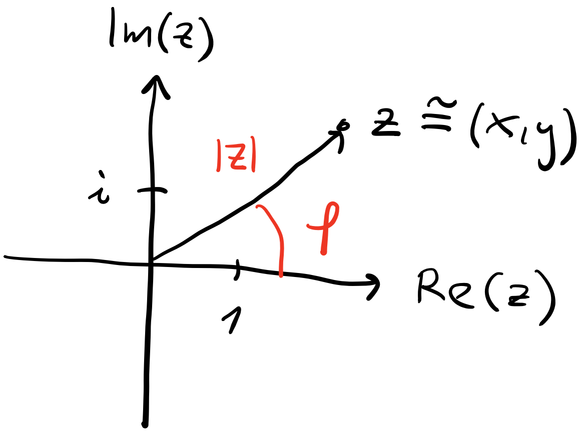
POLARNI ZAPIS KOMPL. ŠTEVILA



POLARNI ZAPIS JE NAČIN, KAKO PODATI KOMPL. ŠTEVILO:

$|z| = \sqrt{x^2+y^2}$ ABS. VREDNOST

$\varphi = \mathrm{Arg}(z) = \underbrace{\mathrm{Arctan}\left(\frac{y}{x}\right)}$ ARGUMENT

RAZŠIRJENI ARKUS TANGENS, KI ZAVZAME VREDNOST NA $[0, 2\pi)$.

$x = |z| \cos\varphi$

$y = |z| \sin\varphi$

$$\boxed{z = |z|(\cos\varphi + i\sin\varphi)}$$

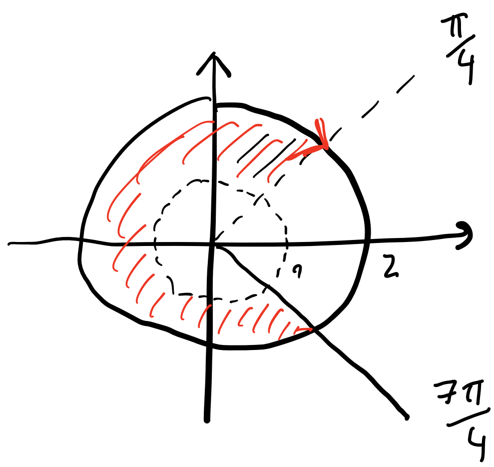
① (a) SKICIRAJ MNOŽICO

$$A = \left\{ z \in \mathbb{C} : \mathrm{Arg}(z) \in \left(\frac{\pi}{4}, \frac{7\pi}{4}\right], \ |z| \in (1,2] \right\}.$$

(b) IZRAČUNAJ PRODUKT:

$$(1+i\sqrt{3})^{42} \cdot (1+i)^{-8}.$$

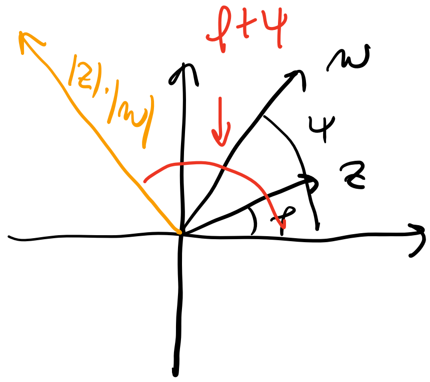
(a)



(b) MNOŽENJE KOMPLEKSNIH ŠTEVIL V POLARNIH KOORDINATAH. VELJA:

$$|z \cdot w| = |z| \cdot |w|$$

$$\mathrm{Arg}(z \cdot w) = \mathrm{Arg}(z) + \mathrm{Arg}(w)$$



t.j. PRI MNOŽENJU SE ARGUMENT SEŠTEVA, ABS. PA MNOŽI.

DOKAZ JE POSLEDICA ADICIJSKIH IZREKOV (GLEJ PREDAVANJA)

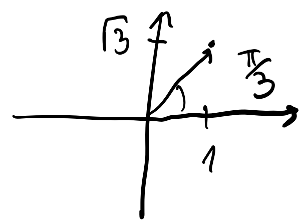
POSLEDICA: DEMOIVROVA FORMULA

$$z^n = |z|^n \cdot (\cos(n\varphi) + i\sin(n\varphi)), \; n \in \mathbb{N}.$$

ZAPIŠIMO ŠTEVILO $1+i\sqrt{3}$ V POLARNIH KOORDINATAH:

$$|1+i\sqrt{3}| = \sqrt{1+3} = 2$$

$$\mathrm{Arg}(1+i\sqrt{3}) = \mathrm{Arctan}\,\frac{\sqrt{3}}{1} = \frac{\pi}{3}$$



VELJA TOREJ:

$$(1+i\sqrt{3}) = 2 \cdot \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

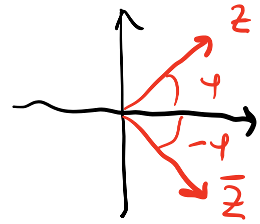
$$(1+i\sqrt{3})^{42} = 2^{42} \cdot \left(\cos\left(42\,\frac{\pi}{3}\right) + i\sin\left(42\,\frac{\pi}{3}\right)\right) =$$

$$= 2^{42} \cdot (\cos(14\pi) + i\sin(14\pi)) =$$

$$= 2^{42} \cdot (1 + i \cdot 0) = 2^{42}.$$

KAKO SE V POLARNIH KOORDINATAH ODRAZI z^{-1}?

$$z^{-1} = \frac{1}{z}\,\frac{/\cdot \bar{z}}{/\cdot \bar{z}} = \frac{\bar{z}}{|z|^2}$$



VEMO: $|\bar{z}| = |z|$, $\mathrm{Arg}(\bar{z}) = -\mathrm{Arg}(z)$

$$|z^{-1}| = \frac{|\bar{z}|}{|z|^2} = \frac{1}{|z|}$$

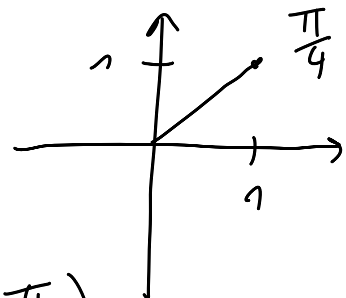
$$\mathrm{Arg}(z^{-1}) = \mathrm{Arg}(\bar{z}) = -\mathrm{Arg}(z)$$

POSLEDICA: $z^{-n} = |z|^{-n}\left(\cos(-n\varphi) + i\sin(-n\varphi)\right) =$

$$\Rightarrow \boxed{z^{-n} = |z|^{-n}\left(\cos(n\varphi) - i\sin(n\varphi)\right)}$$

TO UPORABIMO ZA DRUGI FAKTOR:

$$|1+i| = \sqrt{1+1} = \sqrt{2}$$



$$\arg(1+i) = \frac{\pi}{4}$$

$$(1+i) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$(1+i)^{-8} = \left(\sqrt{2}\right)^{-8} \cdot \left(\cos\left(-8\cdot\frac{\pi}{4}\right) + i\sin\left(-8\frac{\pi}{4}\right)\right)$$

$$= \frac{1}{2^4} \cdot \left(\cos(2\pi) - i\sin(2\pi)\right) =$$

$$= \frac{1}{2^4}$$

SKLEP: $(1+i\sqrt{3})^{42} \cdot (1+i)^{-8} = 2^{42} \cdot \frac{1}{2^4} = \underline{\underline{2^{38}}}$

2 REŠI ENAČBO V OBSEGU KOMPLEKSNIH ŠTEVIL.

(a) $z^2 \bar{z} = 1+i$.

(b) $z^n = \bar{z}$, $n \geq 2$.

IDEJA: KO SMO V PRETEKLOSTI REŠEVALI $\mathbb{C}$- ENAČBE, SMO PRIMERJALI Re IN Im - DELA, TER DOBILI $\mathbb{R}$- SISTEM. SEDAJ BOMO TAK SISTEM DOBILI S PRIMERJAVO ABS. VREDNOSTI IN ARGUMENTA.

(a) $|1+i| = \sqrt{2}$, $\mathrm{Arg}(1+i) = \frac{\pi}{4}$

$1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$|z^2 \cdot \bar{z}| = |z|^2 \cdot |\bar{z}| = |z|^3$

$\mathrm{Arg}(z^2 \cdot \bar{z}) = \mathrm{Arg}(z) + \mathrm{Arg}(z) + \mathrm{Arg}(\bar{z}) =$

$= 2\,\mathrm{Arg}(z) - \mathrm{Arg}(z) = \mathrm{Arg}(z)$

$z^2 \cdot \bar{z} = |z|^3 \cdot \left(\cos(\mathrm{Arg}(z)) + i\sin(\mathrm{Arg}(z))\right) =$

$= \sqrt{2} \cdot \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 1+i$

OD TOD SKLEPAMO:

$|z|^3 = \sqrt{2}$

$\mathrm{Arg}(z) = \frac{\pi}{4} + 2k\pi,\ k \in \mathbb{Z}$

TOREJ JE: $|z| = \sqrt[6]{2}$

$\mathrm{Arg}(z) = \frac{\pi}{4} + 2k\pi$

REŠITEV: $z = \sqrt[6]{2}\left(\cos\left(\frac{\pi}{4} + 2k\pi\right) + i\sin\left(\frac{\pi}{4} + 2k\pi\right)\right)$

$z = \sqrt[6]{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$

OPOMBA: POLARNO NOTACIJO SE DA POENOSTAVITI Z UPORABO t.i. EULERJEVE FORMULE:

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

$$z = |z| \cdot \underbrace{e^{i \cdot Arg(z)}}$$

TA KOMPLEKSNA FUNKCIJA
SE OBNAŠA NA OBIČAJEN NAČIN.
NPR.

$$z \cdot w = |z| \cdot |w| \cdot e^{i\varphi} \cdot e^{i\psi} = |z| \cdot |w| e^{i(\varphi+\psi)}$$

$\varphi = Arg(z)$, $\psi = Arg(w)$

POZOR: BODI PREVIDEN

$$e^{i\varphi} = 1 \Rightarrow \varphi = 0 + 2k\pi$$

(b) $z^n = \bar{z}$, $n \geq 2$.

$$z^n = |z|^n \cdot e^{i\,n\,Arg(z)} = |z|^n e^{in\varphi}$$

$$\bar{z} = |z| \cdot e^{-i\varphi}$$

PRIMERJAMO ABS. VREDNOST IN ARGUMENT:

$$|z|^n = |z|$$

$$n\varphi = -\varphi + 2\pi k, \quad k \in \mathbb{Z}.$$

OD TOD SKLEPAMO:

$$\sim \left(|z|^{n-1} = 1 \Rightarrow |z| = 1\right) \quad \text{ALI} \quad |z| = 0.$$

$$\sim \quad (n+1)\varphi = 2k\pi$$

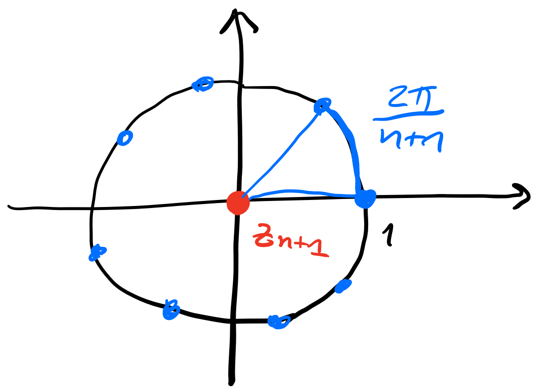
$$\varphi = \frac{2k\pi}{n+1}$$

ČE SE OMEJIMO NA $\varphi \in [0, 2\pi)$ NAM TO PONUDI NASLEDNJE REŠITVE:

$$\varphi_0 = 0, \quad \varphi_1 = \frac{2\pi}{n+1}, \quad \dots, \quad \varphi_n = \frac{n}{n+1} \cdot 2\pi$$

TOREJ IMA ENAČBA $n+2$-REŠITEV:

$$z_{n+1} = 0, \quad z_k = e^{i\frac{2\pi k}{n+1}}, \quad k \in \{0, \dots, n\}$$



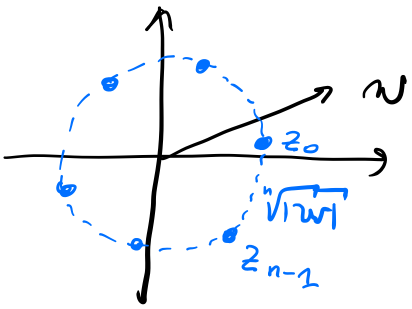
VPRAŠANJE: MORDA STE ŽE SLIŠALI ZA OSNOVNI IZREK ALGEBRE, KI PRAVI: KOMPLEKSEN POLINOM STOPNJE n IMA n NIČEL (ŠTETIH Z VEČKRATNOSTMI).

MI SMO ZA $z^n = \bar{z} \Leftrightarrow z^n - \bar{z} = 0$ DOBILI $(n+2)$-REŠITEV. ALI JE TO POMOTA?

NE. TO NI KOMPLEKSEN POLINOM OZ. POLINOMSKA ENAČBA, KER V NJEJ NASTOPA $\bar{z}$. SICER PA BI RES DOBILI n REŠITEV.

IZ ZGORNJEGA POSTOPKA DOBIMO METODO:

$$\boxed{\begin{aligned} z^n &= w = |w| \cdot e^{i\varphi} \\ z_k &= \sqrt[n]{|w|} \cdot e^{i\left(\frac{\varphi}{n} + \frac{2k\pi}{n}\right)} \end{aligned}}, \quad k \in \{0, 1, \ldots, n-1\}$$



DOBIMO n REŠITEV, KI SO NA KROŽNICI S POLMEROM $\sqrt[n]{|w|}$. PRVA IMA ARGUMENT $\frac{1}{n} \cdot \operatorname{Arg}(w)$, OSTALE PA SO "ENAKOMERNO RAZPOREJENE PO KROŽNICI".

③ REŠI ENAČBE:

(a) $z^4 = 1$.

(b) $z^3 = -1$.

(c) $z^4 = -8 + 8\sqrt{3}\,i$.

V PRIMERIH (a) IN (b) PRIMERJAJ REŠITEV S KLASIČNIM RAZCEPOM

(a) $z^4 = 1$

OBIČAJEN RAZCEP:

$$z^4 - 1 = 0$$

$$(z^2 - 1)(z^2 + 1) = 0$$

$$(z-1)(z+1)(z-i)(z+i) = 0$$

$$\boxed{z_{1,2} = \pm 1} \quad \boxed{z_{3,4} = \pm i}$$

REŠITEV V POLARNIH KOORDINATAH:

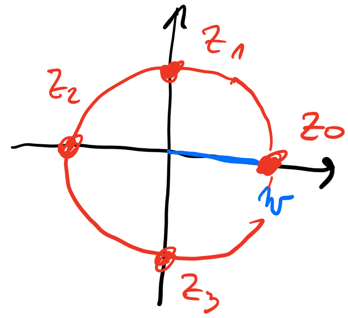
$$w = 1, \quad |w| = 1, \quad \mathrm{Arg}(w) = 0, \quad n = 4,$$

$$z_k = \sqrt[4]{1} \cdot e^{i\left(\frac{0+2k\pi}{4}\right)} = e^{i\frac{k\pi}{2}}, \quad k \in \{0,1,2,3\}.$$

TO SO NASLEDNJE REŠITVE:

$$z_0 = e^{i\cdot 0} = 1$$

$$z_1 = e^{i\frac{\pi}{2}} = \underbrace{\cos\frac{\pi}{2}}_{=0} + i\underbrace{\sin\frac{\pi}{2}}_{=1} = i$$



$$z_2 = e^{i\pi} = \cos\pi + i\sin\pi = -1$$

$$z_3 = e^{i\frac{3\pi}{2}} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i$$

(b) $z^3 = -1$

OBIČAJEN RAZCEP:

$$z^3 + 1 = 0$$

$$(z+1)(z^2 - z + 1) = 0$$

$$z_1 = -1, \quad z_{2,3} = \frac{+1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{-3}}{2} =$$

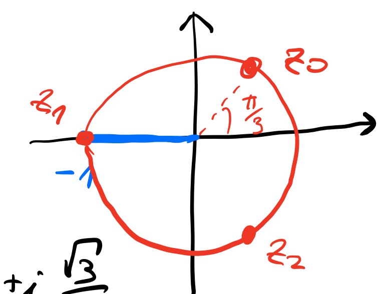
$$= \frac{1 \pm i\sqrt{3}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

S POLARNIMI KOORDINATAMI:

$$w = -1, \quad |w| = 1, \quad \mathrm{Arg}(w) = \pi, \quad n = 3$$

$$z_k = \sqrt[3]{1}\, e^{i\left(\frac{\pi + 2k\pi}{3}\right)} =$$

$$= e^{i\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right)}, \quad k \in \{0,1,2\}.$$



$$z_0 = e^{i\frac{\pi}{3}} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2},$$

$$z_1 = e^{i\pi} = -1$$

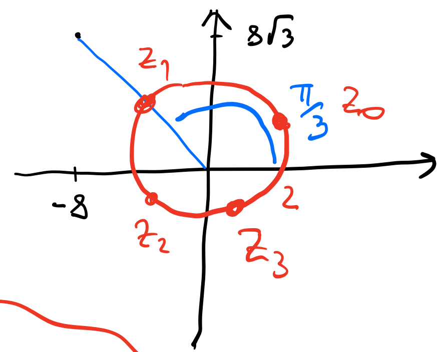
$$z_2 = e^{i\frac{5\pi}{3}} = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

(c) $z^4 = -8 + 8\sqrt{3}\,i = w$

$$|w| = \sqrt{64 + 64 \cdot 3} = 8 \cdot 2 = 16$$

$$\operatorname{Arg}(w) = \operatorname{Arctan} \frac{8\sqrt{3}}{-8} = \operatorname{Arctan}\left(-\sqrt{3}\right) = \frac{2\pi}{3}$$

$$z_k = \sqrt[4]{16} \cdot e^{i\left(\frac{\frac{2\pi}{3} + 2k\pi}{4}\right)}, \quad k \in \{0,1,2,3\}$$



$$z_k = 2\, e^{i\left(\frac{\pi}{6} + \frac{k\pi}{2}\right)}, \quad k \in \{0,1,2,3\}$$

$$z_0 = 2 \cdot e^{i\frac{\pi}{6}} = 2\left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right) = \sqrt{3} + i$$

$$z_1 = 2 \cdot e^{i\frac{2\pi}{3}} = 2\left(\cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3}\right) = -1 + i\sqrt{3}$$

$$z_2 = -z_0 = -\sqrt{3} - i$$

$$z_3 = -z_1 = 1 - i\sqrt{3}$$

KVADRATNA ENAČBA V $\mathbb{C}$

V SŠ SE NAUČIŠ EKSPLICITNO POISKATI REŠITEV:

$$ax^2+bx+c=0\ ;\quad a,b,c\in\mathbb{R}$$

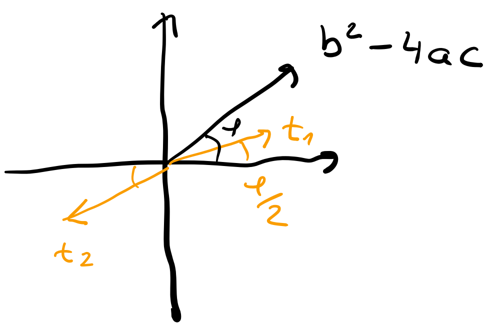
$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

ZAKAJ TA FORMULA DRŽI? KER VELJA:

$$a(x-x_1)(x-x_2)=ax^2+bx+c=0.$$

VPRAŠANJE: ALI SE DA TO FORMULO UPORABITI V PRIMERU, KO $a,b,c\in\mathbb{C}$?

ODGOVOR: DA, A MORAMO NAMESTO $\sqrt{b^2-4ac}$ UPORABITI REŠITVI ENAČBE $t^2=b^2-4ac$.



KER GRE ZA NASPROTNI ŠTEVILI, LAHKO ŠE VEDNO PIŠEMO

$$t_{1,2}=\pm\sqrt{b^2-4ac}$$

AMPAK SE ZAVEDAMO, DA $t_{1,2}$ NISTA NUJNO LE V $\mathbb{R}$ ALI $i\mathbb{R}$.

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① $z^2-(1-4i)z-5-5i=0$

POIŠČI REŠITVI KVADRATNE ENAČBE.

$$a=1,\quad b=-(1-4i),\quad c=-5-5i$$

$$D=b^2-4ac=(1-4i)^2+4(5+5i)=$$

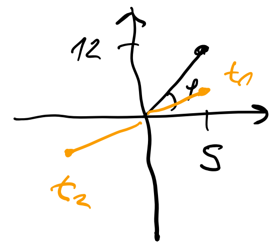
$$=1-8i+16i^2+20+20i=5+12i$$

REŠITVI BOSTA:

$$z_{1,2} = \frac{1-4i \pm \sqrt{5+12i}}{2} = \frac{1-4i+t_{1,2}}{2},$$

KJER $t_{1,2}$ REŠITA $\boxed{t^2 = 5+12i}$.

$$|5+12i| = \sqrt{25+144} = \sqrt{169} = 13$$



$$\varphi = Arg(5+12i) = Arc\tan\left(\frac{12}{5}\right) \in [0, \tfrac{\pi}{2}].$$

$$t_{1,2} = \pm\sqrt{13}\, e^{i\frac{\varphi}{2}} = \sqrt{13}\, e^{i(\frac{\varphi}{2}+\pi)}$$

NASPROTNI ŠTEVILI

$n=2$

OD ZADNJIČ

$z^n = w$

$\varphi = Arg(w)$

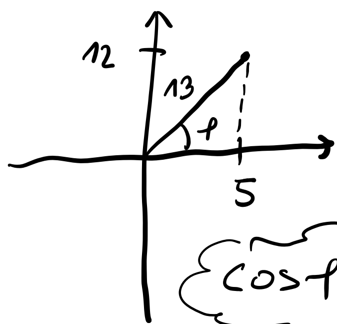
$z_k = \sqrt[n]{|w|}\, e^{i\left(\frac{\varphi}{n}+\frac{2k\pi}{n}\right)}$

$k \in \{0, 1, \dots, n-1\}$

$$t_{1,2} = \pm\sqrt{13}\left(\cos\frac{\varphi}{2} + i\sin\frac{\varphi}{2}\right)$$

$$\cos\frac{\varphi}{2} = \pm\sqrt{\frac{1+\cos\varphi}{2}}$$

$$\sin\frac{\varphi}{2} = \pm\sqrt{\frac{1-\cos\varphi}{2}}$$



$$\cos\varphi = \frac{5}{13}$$

ZAKAJ $\pm$?

PREDZNAK SE IZBERE GLEDE NA TO, V KATEREM KVADRANTU JE $\varphi/2$.

V NAŠEM PRIMERU

$\cos\frac{\varphi}{2} > 0$

$\sin\frac{\varphi}{2} > 0$, $\quad \frac{\varphi}{2} \in [0, \frac{\pi}{4}]$.

$$\cos\frac{\varphi}{2} = +\sqrt{\frac{1+\frac{5}{13}}{2}} = \sqrt{\frac{18}{2\cdot 13}} = \sqrt{\frac{9}{13}} = \frac{3}{\sqrt{13}}$$

$$\sin\frac{\varphi}{2} = +\sqrt{\frac{1-\frac{5}{13}}{2}} = \sqrt{\frac{8}{2\cdot 13}} = \sqrt{\frac{4}{13}} = \frac{2}{\sqrt{13}}$$

$$t_{1,2} = \pm\sqrt{13}\left(\frac{3}{\sqrt{13}} + i\frac{2}{\sqrt{13}}\right) = \pm(3+2i)$$

$$z_{1,2} = \frac{1-4i \pm (3+2i)}{2} \Rightarrow \boxed{z_1 = 2-i}, \boxed{z_2 = -1-3i}$$