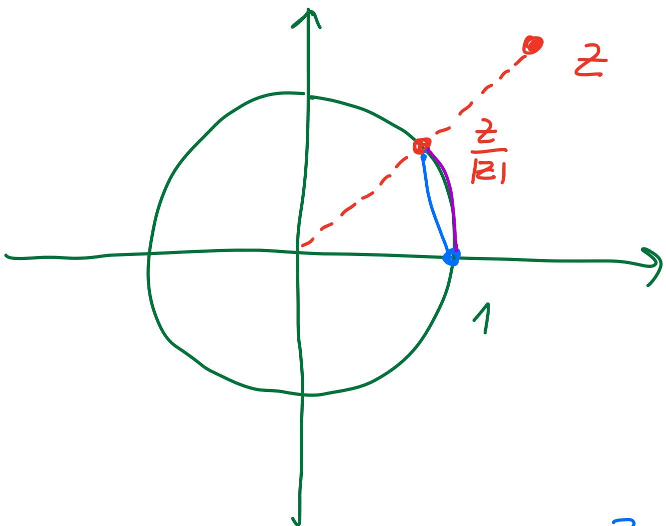
① DOKAŽI, DA ZA $z \in \mathbb{C} \setminus \{0\}$ VELJA

$$\left| \frac{z}{|z|} - 1 \right| \leq \arg(z)$$

KDAJ VELJA ENAKOST?

R: KAJ POMENI ŠTEVILO $\frac{z}{|z|}$. TO JE NORMIRANI VEKTOR V SMERI z.



MODRA RAZDALJA JE ENAKA $\left| \frac{z}{|z|} - 1 \right|$.
VIJOLIČNI DEL LOKA JE DOLG $\mathrm{Arg}(z)$, ČE GA MERIMO V RADIANIH.

NEENAKOST TOREJ POVE:

DOLŽINA TETIVE $\leq$ DOLŽINA LOKA

ENAKOST JE IZPOLNJENA LE ZA $\varphi = 0$ OZ. KO JE $z \in \mathbb{R}_+$.

② $a \in \mathbb{C} \setminus \{0\}$, $b \in \mathbb{C}$.

DOKAŽI, DA IMA $z^2 - 2az + b = 0$ REŠITVI NA ENOTSKI KROŽNICI NATANKO TEDAJ, KO $|a| \leq 1$, $|b| = 1$ IN $\mathrm{Arg}(b) = 2\,\mathrm{Arg}(a)$.

($\Leftarrow$) $\quad |a| \leq 1, |b| = 1, \mathrm{Arg}(b) = 2 \cdot \mathrm{Arg}(a) \Rightarrow |z_{1,2}| = 1$

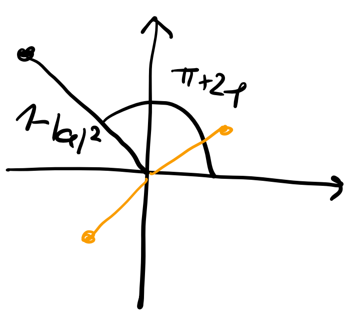
VEMO:
$$z_{1,2} = \frac{2a \pm \sqrt{4a^2 - 4b}}{2} = a \pm \sqrt{a^2 - b} = a + t_{1,2},$$
KJER STA $t_{1,2}$ REŠITVI ENAČBE $t^2 = a^2 - b$.

PIŠIMO $\varphi = \mathrm{Arg}(a)$, $\mathrm{Arg}(b) = 2\varphi$, TER IZRAZIMO:
$$a^2 - b \underset{\substack{\uparrow \\ \text{POLARNI} \\ \text{ZAPIS}}}{=} (|a| e^{i\varphi})^2 - |b| e^{i2\varphi} \underset{\substack{\uparrow \\ |b|=1}}{=} (|a|^2 - 1) e^{2i\varphi}$$

KER JE $|a| \leq 1$, JE $|a|^2 - 1 \leq 0$, ZATO ZAPIS POMNOŽIMO Z (-1):
$$a^2 - b = (-1)(1 - |a|^2) e^{2i\varphi} = e^{i\pi} \cdot (1 - |a|^2) e^{2i\varphi} = (1 - |a|^2) e^{i(2\varphi + \pi)}$$

TO JE POLARNI ZAPIS ŠTEVILA, KI SE POJAVI NA DESNI STRANI ENAČBE $t^2 = a^2 - b$. SEDAJ LOČIMO PRIMERA:

(i) DENIMO DA JE $|a| < 1$.



POTEM STA REŠITVI $t_{1,2}$ ENAKI:
$$t_1 = \sqrt{1 - |a|^2}\, e^{i\left(\frac{2\varphi + \pi}{2}\right)} = i\sqrt{1 - |a|^2}\, e^{i\varphi}$$
$$t_2 = -t_1 = -i\sqrt{1 - |a|^2}\, e^{i\varphi}$$

TO POMENI, DA VELJA
$$z_{1,2} = a \pm i\sqrt{1 - |a|^2}\, e^{i\varphi}$$
$$= |a| e^{i\varphi} \pm i\sqrt{1 - |a|^2}\, e^{i\varphi}$$
$$= (|a| \pm i\sqrt{1 - |a|^2})\, e^{i\varphi}$$

IZRAČUNAJMO ABSOLUTNO VREDNOST

$$|z_{1,2}| = \left| |a| \pm i\sqrt{1-|a|^2} \right| \cdot \underbrace{|e^{i\varphi}|}_{|\cos\varphi + i\sin\varphi| = 1}$$

$$= \sqrt{|a|^2 + \left(\pm\sqrt{1-|a|^2}\right)^2} = \sqrt{|a|^2 + 1 - |a|^2} = \underline{\underline{1}}$$

TOREJ VELJA ŽELJENI SKLEP.

(ii) DENIMO, DA $|a| = 1$.

POTEM JE $a^2 - b = (1-|a|^2)e^{i\varphi} = 0$ OZ. $t_{1,2} = 0$.

TOREJ JE $z_{1,2} = a$ DVOJNA NIČLA IN $|z_{1,2}| = |a| = 1$.

$(\Rightarrow)$ $|z_{1,2}| = 1 \Rightarrow |a| \leq 1,\ |b| = 1,\ \mathrm{Arg}(b) = 2\,\mathrm{Arg}(a)$

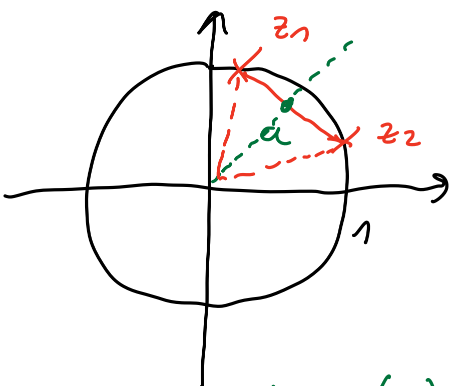
VEMO: $z^2 - 2az + b = (z - z_1)(z - z_2) = z^2 - (z_1 + z_2)z + z_1 z_2 = 0$

PO VIETOVIH PRAVILIH TOREJ VELJA:

$$z_1 + z_2 = 2a \quad \text{IN} \quad z_1 \cdot z_2 = b.$$

DRUGA ENAKOST POVE: $|b| = |z_1| \cdot |z_2| = 1 \cdot 1 = 1.$

IZ PRVE PA IZVEMO, DA JE $a = \dfrac{z_1 + z_2}{2}$. SKICA:

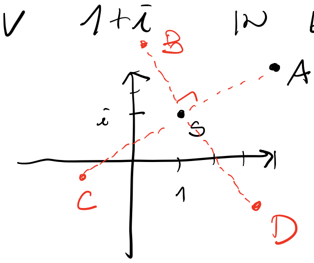


a JE RAZPOLOVIŠČE TETIVE SKOZI z_1 IN z_2, ZATO JE $|a| \leq 1$.

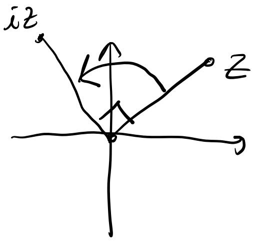
ŠE VEČ, DAWICA SKOZI IZHODIŠČE IN a JE SIMETRALA KOTOV $\mathrm{Arg}(z_1)$ IN $\mathrm{Arg}(z_2)$. TOREJ IMAMO:

$$\mathrm{Arg}(a) = \frac{\mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)}{2} = \frac{1}{2}\mathrm{Arg}(z_1 \cdot z_2) = \frac{1}{2}\mathrm{Arg}(b)$$

③ DOLOČI OGLIŠČA KVADRATA, KI IMA SREDIŠČE V $1+i$ IN ENO OGLIŠČE V $4+3i$.



IDEJA: ROTIRATI MORAMO ZA $90°$.
V $\mathbb{C}$ TO ROTACIJO PODAJA RAVNO
MNOŽENJE Z i.
$|i \cdot z| = |z|$ in $\operatorname{Arg}(iz) = \operatorname{Arg}(i) + \operatorname{Arg}(z) = \frac{\pi}{2} + \operatorname{Arg}(z)$



NAJ BO $a = 4+3i$ IN $s = 1+i$.
TEDAJ b, KI IDENTIFICIRA OGLIŠČE B
DOBIMO Z IZRAZOM:

$$b = s + i(a-s)$$

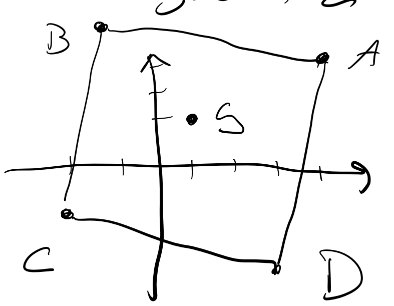
ČE ŽELIMO PRITI OD O DO b,
GREMO NAJPREJ DO s, NATO
PA PO VEKTORJU $\overrightarrow{SA}$ ZAROTIRANEM
ZA $90°$ V POZITIVNI SMERI.

$$b = (1+i) + i\left(4+3i - (1+i)\right) =$$
$$= 1+i+3i-2 = \underline{-1+4i}$$

PODOBNO VELJA:

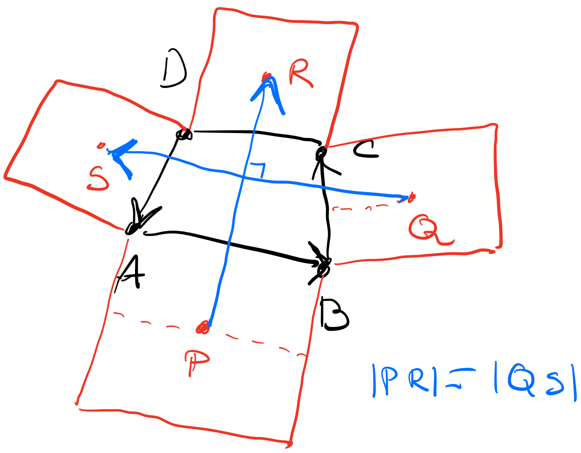
$$c = s - (a-s)$$
$$d = s - i(a-s)$$
$$c = (1+i) - (4+3i-1-i) =$$
$$= 1+i-3-2i = \underline{-2-i}$$

$$d = (1+i) - i(3+2i) =$$
$$= 1+i-3i+2 = \underline{3-2i}$$



DOKAŽI IZREK:

NAJ BODO A, B, C IN D OGLIŠČA ŠTIRIKOTNIKA. NAD VSAKO STRANICO NARIŠEMO KVADRAT IN NJIHOVA SREDIŠČA OZNAČIMO Z ZAPOREDOMA Z P, Q, R IN S. DOKAŽI, DA JE $PR \perp QS$ IN $|PR| = |QS|$.



IZZIV: POSKUSI TA IZREK DOKAZATI BREZ UPORABE KOMPLEKSNIH ŠTEVIL.

OZNAČIMO Z a, b, c, d, p, q, r, s KOMPLEKSNA ŠTEVILA, KI IDENTIFICIRAJO OGLIŠČA OZ. SREDIŠČA.

TEDAJ VELJA:

$$p = a + \frac{1}{2}(b-a) + \frac{1}{2}(b-a)\cdot(-i)$$

ČE ŽELIMO PRITI DO p, GREMO NAJPREJ V a, NATO SE POMAKNEMO ZA POLOVICO VEKTORJA $\overrightarrow{AB}$, NATO PA ŠE ZA ISTI VEKTOR ZAROTIRAN ZA $90°$ V NEGATIVNO SMER.

$$\boxed{p = \frac{1}{2}(a+b) - \frac{i}{2}(b-a)}$$

NA PODOBEN NAČIN DOBIMO TUDI NASLEDNJE ZVEZE.

$$\boxed{\begin{aligned} q &= \tfrac{1}{2}(b+c) - \tfrac{i}{2}(c-b) \\ r &= \tfrac{1}{2}(c+d) - \tfrac{i}{2}(d-c) \\ s &= \tfrac{1}{2}(d+a) - \tfrac{i}{2}(a-d) \end{aligned}}$$

KAJ ŽELIMO DOKAZATI?

$$i \cdot (r-p) = s-q$$

ČE ZAROTIRAMO $\overrightarrow{PR}$ V POZITIVNO SMER ZA 90°, DOBIMO $\overrightarrow{QS}$. TO NAM POTRDI OBE DOMNEVI t.j.:

$PR \perp QS$ IN $|PR| = |QS|$

$$i \cdot \left(\tfrac{1}{2}(c+d) - \tfrac{i}{2}(d-c) - \tfrac{1}{2}(a+b) + \tfrac{i}{2}(b-a) \right) =$$

$$= \tfrac{1}{2}(a+d) - \tfrac{i}{2}(a-d) - \tfrac{1}{2}(b+c) + \tfrac{i}{2}(c-b) \quad / \cdot 2$$

$$i \cdot (c+d-a-b) + (d-c-b+a) =$$

$$= (a+d-b-c) + i(-a+d+c-b)$$

✓

□

⑤ IZPELJI ADICIJSKI IZREK ZA $\cos(n\varphi)$ IN $\sin(n\varphi)$, KO $n=3$ IN $n=4$.

R: RADI BI DOBILI ZAKLJUČENO FORMULO, KI BO VSEBOVALA LE $\cos(\varphi)$ IN $\sin(\varphi)$. TO LAHKO STORIMO DIREKTNO NPR.

$$\cos(2\varphi) = \cos^2\varphi - \sin^2\varphi$$
$$\sin(2\varphi) = 2\sin\varphi\cos\varphi$$
$$\begin{aligned}\cos(3\varphi) &= \cos(2\varphi+\varphi)\\ &= \cos(2\varphi)\cos(\varphi) - \sin(2\varphi)\sin(\varphi)\\ &= \cos^3\varphi - \cos\varphi\sin^2\varphi - 2\sin^2\varphi\cos\varphi\\ &= \cos^3\varphi - 3\cos\varphi\sin^2\varphi\end{aligned}$$
$$\vdots$$

ALTERNATIVA JE UPORABA DEMOIROVE FORMULE.

$$e^{in\varphi} = \cos(n\varphi) + i\sin(n\varphi)$$

ZA $n=3$ DOBIMO:

$$(\cos\varphi + i\sin\varphi)^3 = \cos(3\varphi) + i\sin(3\varphi)$$
$$\|$$
$$\cos^3\varphi + 3i\cos^2\varphi\sin\varphi - 3\cos\varphi\sin^2\varphi - i\sin^3\varphi$$

SKLEP: $\cos(3\varphi) = \cos^3\varphi - 3\cos\varphi\sin^2\varphi$

$\sin(3\varphi) = 3\cos^2\varphi\sin\varphi - \sin^3\varphi$

ZA $n=4$ DOBIMO:

$$\cos(4\varphi)+i\sin(4\varphi)=(\cos\varphi+i\sin\varphi)^4=$$

$$=\cos^4\varphi+4i\cos^3\varphi\sin\varphi+$$
$$+6\cos^2\varphi(i\sin\varphi)^2+4\cos\varphi(i\sin\varphi)^3$$
$$+(i\sin\varphi)^4=$$

$$=\cos^4\varphi+4i\cos^3\varphi\sin\varphi-6\cos^2\varphi\sin^2\varphi$$
$$-4i\cos\varphi\sin^3\varphi+\sin^4\varphi$$

SKLEP:

$$\cos(4\varphi)=\cos^4\varphi-6\cos^2\varphi\sin^2\varphi+\sin^4\varphi$$
$$\sin(4\varphi)=4\cos^3\varphi\sin\varphi-4\cos\varphi\sin^3\varphi.$$

(6) ZA POLJUBEN $\varphi\in(0,2\pi)$ IN $n\in\mathbb{N}$ IZRAČUNAJ:

$$C_n=\sum_{k=0}^{n}\cos(k\varphi)=1+\cos(\varphi)+\ldots+\cos(n\varphi)$$
$$S_n=\sum_{k=1}^{n}\sin(k\varphi)=\sin(\varphi)+\sin(2\varphi)+\ldots+\sin(n\varphi)$$

R: PO DE MOIROVI FORMULI VELJA:

$$C_n+iS_n=(1+\cos(\varphi)+\ldots+\cos(n\varphi))+i(\sin(\varphi)+\ldots+\sin(n\varphi))$$
$$=1+(\cos\varphi+i\sin\varphi)+\ldots+(\cos(n\varphi)+i\sin(n\varphi))=$$
$$=1+e^{i\varphi}+e^{2i\varphi}+\ldots+e^{in\varphi}$$

TOREJ IMAMO OPRAVKA Z $(n+1)$ ČLENI GEOMETRIJ. ZAPOREDJA S ČLENOM $q=e^{i\varphi}$.

KER $\varphi \neq 0$ IN $\varphi \neq 2\pi$, JE $q \neq 1$. TOREJ VELJA:

$$C_n + i S_n = \frac{e^{i(n+1)\varphi}-1}{e^{i\varphi}-1} \cdot \frac{e^{-i\varphi}-1}{e^{-i\varphi}-1}$$

$$= \frac{e^{in\varphi}-e^{-i\varphi}-e^{i(n+1)\varphi}+1}{2-e^{i\varphi}-e^{-i\varphi}} =$$

$$= \frac{\cos(n\varphi)+i\sin(n\varphi)-\cos(\varphi)+i\sin(\varphi)-\cos((n+1)\varphi)-i\sin((n+1)\varphi)+1}{2-\cos(\varphi)-\cancel{i\sin\varphi}-\cos\varphi+\cancel{i\sin(\varphi)}}$$

OD TOD SLEDI:

$$C_n = \frac{\cos(n\varphi)-\cos\varphi-\cos((n+1)\varphi)+1}{2-2\cos(\varphi)}$$

$$S_n = \frac{\sin(n\varphi)+\sin(\varphi)-\sin((n+1)\varphi)}{2-2\cos(\varphi)}$$

SEDAJ UPOŠTEVAMO:

$$\cos(n\varphi)-\cos((n+1)\varphi) = -2\sin\left(\frac{2n+1}{2}\varphi\right)\sin\left(-\frac{\varphi}{2}\right)$$
$$= 2\sin\left(\left(n+\frac{1}{2}\right)\varphi\right)\sin\left(\frac{\varphi}{2}\right)$$

$$\sin(n\varphi)-\sin((n+1)\varphi) = 2\cos\left(\frac{2n+1}{2}\varphi\right)\sin\left(-\frac{\varphi}{2}\right) =$$
$$= -2\cos\left(\left(n+\frac{1}{2}\right)\varphi\right)\sin\left(\frac{\varphi}{2}\right)$$

$$2-2\cos\varphi = 4\cdot\frac{1-\cos\varphi}{2} = 4\cdot\sin^2\frac{\varphi}{2}$$

DOBIMO:

$$C_n = \frac{1}{2} + \frac{2\sin\left(\left(n+\frac{1}{2}\right)\varphi\right)\sin\frac{\varphi}{2}}{4\sin^2\frac{\varphi}{2}} = \frac{1}{2}+\frac{1}{2}\,\frac{\sin\left(\left(n+\frac{1}{2}\right)\varphi\right)}{\sin\varphi}$$

$$S_n = \frac{2\sin\left(\frac{\varphi}{2}\right)\cdot\cos\left(\frac{\varphi}{2}\right)}{4\sin^2\frac{\varphi}{2}} - \frac{2\cos\left(\left(n+\frac{1}{2}\right)\varphi\right)\sin\left(\frac{\varphi}{2}\right)}{4\sin^2\frac{\varphi}{2}} = \frac{1}{2}\operatorname{ctg}\varphi - \frac{1}{2}\,\frac{\cos\left(\left(n+\frac{1}{2}\right)\varphi\right)}{\sin\frac{\varphi}{2}}$$