# Exponentiation and factorials

1. a) In how many ways can 4 different cereals be displayed on a shelf?
   b) In how many ways can 5 people be arranged in a circle?

2. a) A sports league has four teams. If we want each team to play with each other team exactly once per season, how many matches should there be in total in a season?
   b) Two more teams join the league. How many matches do there need to be per season now?
   c) If instead of sports matches between two out of six teams, we are counting the number of ways of choosing two balls out of a bag of six, then the number is the same. How many ways are there of choosing three out of the six balls? What difference does it make if we count not only which three balls are picked, but also the order we pick them?

3. Expand the expression $(x+y)^4$ (that is, rewrite it as a sum of coefficients of the form $cx^ay^b$, where $a$, $b$ and $c$ are numbers, with no brackets left). Can you guess what the expanded form of $(x+y)^6$ is now without doing any counting?

4. (Bernoulli trials) Imagine a random experiment or game where the probability of success is $\frac{1}{n}$ every time; for example, rolling an $n$-sided die and seeing if it lands on a particular side, or picking the top card from a freshly-shuffled deck of $n$ different cards, where success is when a particular card is the one on top.

   a) If we try the experiment or game $n$ times, what is the probability of winning $k$ of those times? Hint: start with $n = 2$, for example flipping an ordinary coin, where success is getting heads; if we flip the coin twice, what is the probability of getting (i) no heads ($k = 0$), (ii) one head and one tail, either way around ($k = 1$), or (iii) heads both times ($k = 2$)?
   b) In particular, what is the probability of winning (a random game with $\frac{1}{n}$ probability of success) zero times after $n$ tries?

5. The expression

$$\left(1 - \frac{1}{n}\right)^n$$

gets closer and closer to $1/e$ as $n$ gets bigger and bigger (in mathematics we say the expression *tends to* $1/e$ *as* $n$ *approaches infinity*). Using a calculator or the calculator app on your phone or other computer if you have one, or using Google, another search engine, or a website such as Wolfram Alpha if you prefer, try to get a good approximation of $1/e$. Test a few different values of $n$ and see how much difference the size of $n$ makes to how close your estimate is. The exact value of $1/e$ is 0.367879441171442 to 15 significant figures.

The value of $e$ itself to 15 significant figures is 2.71828182845904. The sum

$$\sum_{k=0}^{n} \frac{1}{k!},$$

of the first $n+1$ factorials including zero, tends to $e$ as $n$ approaches infinity. Use your calculator to add successive terms of the sum, and see how quickly the total seems to approach $e$.

6. Imagine you have a savings account with a bank that offers 100% interest per year, paid once at the end of the year. In other words, if you put one euro into the account, after a year you will have two euros. If, instead, they give you 50% interest twice a year, then after six months your euro will have increased to €1.50, and after another six months, you will get 50% of *this* as interest, and will end the year with €2.25. By what factor will your money increase in a year if the bank spreads the 100% interest over four payments instead (that is, 25% interest paid every three months)? What about monthly payments? Write down a formula for what factor your account balance has multiplied by at the end of the year if the interest is paid in $n$ instalments of a fraction that adds up to 100% over the year. How much would you have after a year of weekly interest payments? Daily? What do you notice about these numbers?

7. For any $x$, the sum of the first $n$ terms in the series

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots,$$

which we can write as $\sum_{k=0}^{n} \frac{x^n}{k!}$, tends to $e^x$ as $n$ approaches infinity. Write out an expanded formula for $e^{x+y}$ (where $x$ and $y$ are small enough numbers that if we multiply more than three of them together the result is so small that we can ignore it), and show that it equals $e^x \times e^y$ (we already know this is true thanks to the exponentiation laws).

8. Here are some exponent laws once again:

$$x^a x^b = x^{a+b} \qquad (x^a)^b = x^{ab} \qquad \frac{x^a}{x^b} = x^{a-b} \qquad x^0 = 1$$

See if you can come up with some laws for logarithms based on these. For example, if we let $A = x^a$ and $B = x^b$, then $x^a x^b = x^{a+b}$ gives us $\log_x(AB) = \log_x A + \log_x B$ (by taking the log base $x$ of both sides of the equality).