Complex numbers

1. Write $\dfrac{1+2i}{3+4i}$ in the form $a+bi$.

2. How can the real and imaginary part of a complex number z be written in terms of z and the complex conjugate of z? What about the absolute value? How can you see this geometrically?

3. Consider a complex number z with polar coordinates (r, θ). What are the cartesian coordinates of z? Hence, what are some different ways of writing z?

4. Using the formula $e^{ix} = \sin x + i \cos x$, does complex exponentiation follow the same exponent rules as the real version, i. e. does it hold

$$e^{ix}e^{iy} = e^{i(x+y)}$$

   for any x, $y \in \mathbb{R}$.

5. Write $-1 + i\sqrt{3}$ in polar and exponential form.

6. Find the fourth roots of -4 in the form $a+bi$ and factorise z^4+4 into linear factors. Then write z^4+4 as a product of real quadratic factors

7. Use the exponential form of a complex number to show that for any $n \geq 2$, the sum of the n-th roots of unity is zero.

8. Let $f\colon \mathbb{C} \to \mathbb{C}$ be given as $f(z) = e^z$.

   1. Show that the complex exponential maps vertical lines to circles with the origin as center.
   2. Show that the complex exponential maps horizontal lines to half rays from the origin to infinity.

   Deduce that f preserves angles. Such mappings are called *conformal*.

9. Choose a small complex number (it can be a real number if you want); call it z_0. Using a calculator of your choice, work out the terms of the sequence given by $z_{i+1} = z_i^2 + z_0$. Do you think your z_0 is in the Mandelbrot set, or not?