Matrices (part 2)

1. Find the inverse of the matrix $\begin{pmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{pmatrix}$ two different ways. Use Gaussian elimination and the formula with the adjugate matrix.

   Solution: $\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{pmatrix}$

2. Solve the following system of linear equations

$$\begin{aligned} 3x + 2z &= 1 \\ 2x - 2z &= -1 \\ y + z &= 0 \end{aligned}$$

   two different ways. Use Gaussian elimination and the inverse matrix method.

   Solution: $\left(0, -\frac{1}{2}, \frac{1}{2}\right)$

3. Consider the matrix

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

   a) Multiply the matrix with the vector $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. What does the matrix P represent?

   b) Find the inverse of matrix P.

   Solution: $P^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

4. Find the inverse of the matrix $\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{pmatrix}$.

Solution: $\begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} & 0 \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} & 0 \\ 0 & 0 & \frac{1}{e} \end{pmatrix}$

5. Prove the following result: If A and B are nonsingular (invertible) matrices, then AB is nonsingular and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

6. Find the linear equations that need to be solved to 'balance' the following chemical equation:

$$a\,\mathrm{CO_2} + b\,\mathrm{H_2O} \xrightarrow{light} \mathrm{C_6H_{12}O_6} + c\,\mathrm{O_2}.$$

Solution: $a = b = c = 6$

7. If A^{-1} exists, what is the relationship between $\det(A)$ and $\det(A^{-1})$. Explain your answer.

Solution: $\det(A^{-1}) = \frac{1}{\det(A)}$

8. Let A be an $n \times n$ matrix where n is odd. Suppose also that A is skew symmetric. This means $A^T = -A$. Show that $\det(A) = 0$.

9. In 1867, Charles L. Dodgson gave us an alternative way to calculate the determinant of a 3×3-matrix. Let $M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, then we make a 2×2-matrix M' out of the determinants of the four *connected minors* $\begin{pmatrix} a & b \\ d & e \end{pmatrix}$, $\begin{pmatrix} b & c \\ e & f \end{pmatrix}$, $\begin{pmatrix} d & e \\ g & h \end{pmatrix}$, and $\begin{pmatrix} e & f \\ h & i \end{pmatrix}$. Then the determinant of the matrix M', divided by the middle entry e of the original matrix, is the determinant of M. Prove this always gives the same result as the method of calculating the determinant that we already know.