Rešitve 2. kolokvija

2. KOLOKVIJ
REŠITVE

NALOGA 1

a) $f(x) = \ln(x^3 - 3x^2 + 2x)$

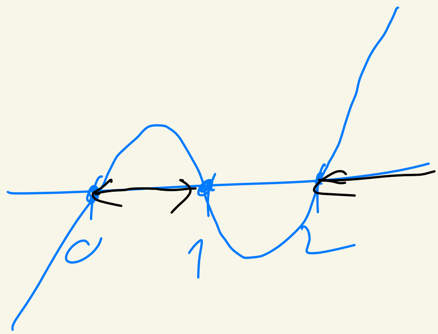
$D_f = ?$

$x^3 - 3x^2 + 2x > 0$

$x(x^2 - 3x + 2) > 0$

$x(x-2)(x-1) > 0$

$x_1 = 0, \; x_2 = 1, \; x_3 = 2$



$\Rightarrow D_f = (0,1) \cup (2, \infty)$

$Z_f = ?$

$g(x) = x^3 - 3x^2 + 2X$

zaloga vrednosti za g za $x \in D_f$ je

množica $(0, \infty) \Rightarrow Z_f = \mathbb{R}$

b) $f(\underbrace{y^2-2y}_{x}) = 1-y$

$\Rightarrow y^2 - 2y - x = 0$

Izrazimo y: $D = 4+4x = 4(1+x)$

$\Rightarrow y_{1,2} = \dfrac{2 \pm 2\sqrt{1+x}}{2} = 1 \pm \sqrt{1+x}$

kad $y \in (-\infty, 1]$ i $x \in [-1, \infty)$

$\Rightarrow y = 1 - \sqrt{1+x}$

$\Rightarrow f(x) = f(y^2-2y) = 1-y = 1-(1-\sqrt{1+x})$

$f(x) = \sqrt{1+x}$

provera:

$f(y^2-2y) = \sqrt{1+y^2-2y} = \sqrt{(y-1)^2} = |\underbrace{y-1}_{\leq 0}| = 1-y$

ZADANIE 2

$$a_n = \ln(2^{n+1}-1) - n\ln 2$$
$$= \ln(2^{n+1}-1) - \ln 2^n$$
$$= \ln\left(\frac{2^{n+1}-1}{2^n}\right) = \ln\left(2-\frac{1}{2^n}\right)$$

a) opaziuj, daje a_n naraścający

Dokaz: $\underline{a_{n+1} \geq a_n}$

$$\ln\left(2-\frac{1}{2^{n+1}}\right) \overset{?}{\geq} \ln\left(2-\frac{1}{2^n}\right)$$

$$2-\frac{1}{2^{n+1}} \overset{?}{\geq} 2-\frac{1}{2^n}$$

$$\frac{1}{2^{n+1}} \overset{?}{\leq} \frac{1}{2^n}$$

$$2^{n+1} \overset{?}{\geq} 2^n$$

$$2 \geq 1 \quad \checkmark$$

b) Dokažimo, da je zaporedje navzgor omejeno z $\ln 2$.

$$a_n \le \ln 2$$

$$\ln\left(2-\frac{1}{2^n}\right) \overset{?}{\le} \ln 2$$

$$2-\frac{1}{2^n} \overset{?}{\le} 2$$

$$\frac{1}{2^n} \ge 0 \quad \checkmark$$

$\Rightarrow$ ker je zaporedje naraščajoče in navzgor omejeno, je konvergentno.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(2-\frac{1}{2^n}\right)$$

$$= \ln\left(\lim_{n\to\infty}\left(2-\frac{1}{2^n}\right)\right) = \underline{\ln 2}.$$

$$\frac{1}{2^n} \to 0$$

c) $\varepsilon = \ln 1.1$

$$|a_n - a| < \varepsilon$$

$$\left|\ln\left(2 - \frac{1}{2^n}\right) - \ln 2\right| < \varepsilon$$

$$\left|\ln\underbrace{\left(1 - \frac{1}{2^{n+1}}\right)}_{<1}\right| < \varepsilon$$

$$-\ln\left(1 - \frac{1}{2^{n+1}}\right) < \varepsilon$$

$$\ln\left(1 - \frac{1}{2^{n+1}}\right) > -\varepsilon = -\ln 1.1$$

$$\ln\left(1 - \frac{1}{2^{n+1}}\right) > -\ln\frac{11}{10}$$

$$\ln\left(1 - \frac{1}{2^{n+1}}\right) > \ln\frac{10}{11}$$

$$\ln\left(1 - \frac{1}{2^{n+1}}\right) > \ln\left(1 - \frac{1}{11}\right)$$

$$1 - \frac{1}{2^{n+1}} > 1 - \frac{1}{11}$$

$$\frac{1}{2^{n+1}} < \frac{1}{11}$$

$$2^{n+1} > 11 \Rightarrow n+1 \geq 4$$

$$n \geq 3$$

$\Rightarrow$ od 3. člena naprej se členi od limite razlikujejo za manj kot ε.

NALOGA 3

a)

$$\lim_{n\to\infty} \frac{\sqrt{n+2}-\sqrt{n}}{n\sqrt{n}-\sqrt{n^3-n}} \cdot \frac{(\sqrt{n+2}+\sqrt{n})(n\sqrt{n}+\sqrt{n^3-n})}{(\sqrt{n+2}+\sqrt{n})(n\sqrt{n}+\sqrt{n^3-n})}$$

$$= \lim_{n\to\infty} \frac{(n+2-n)(n\sqrt{n}+\sqrt{n^3-n})}{(\sqrt{n+2}+\sqrt{n})(n^3-(n^3-n))}$$

$$= \lim_{n\to\infty} \frac{2(n\sqrt{n}+\sqrt{n^3-n})}{(\sqrt{n+2}+\sqrt{n})n} \quad \begin{array}{l} /:n^{\frac{3}{2}} \\ /:n^{\frac{3}{2}} \end{array}$$

$$= \lim_{n\to\infty} \frac{2\left(1+\sqrt{1-\frac{1}{n^2}}\right)}{\sqrt{1+\frac{2}{n}}+\sqrt{1}} = \frac{4}{2} = \underline{\underline{2}}$$

b) $\lim\limits_{n\to\infty}\left(\dfrac{2^n-1}{2^n+5}\right)^{2^n}$

$$= \lim_{n\to\infty}\left(\frac{2^n+5-6}{2^n+5}\right)^{2^n}$$

$$= \lim_{n\to\infty}\underbrace{\left(1+\frac{1}{\frac{2^n+5}{-6}}\right)^{\frac{2^n+5}{(-6)}}}_{e}{}^{\cdot\underbrace{\frac{(-6)}{2^n+5}\cdot 2^n}_{-6}}$$

$$= \underline{\underline{e^{-6}}}$$

NALOGA 4

a) $$\sum_{n=1}^{\infty} \frac{2^{n+1}+5^{n}}{3^{2n}} = \sum_{n=1}^{\infty} \frac{2\cdot 2^{n}+5^{n}}{9^{n}}$$

$$= 2\cdot \sum_{n=1}^{\infty} \left(\frac{2}{9}\right)^{n} + \sum_{n=1}^{\infty} \left(\frac{5}{9}\right)^{n}$$

$$= 2\cdot \frac{2}{9}\left(1+\frac{2}{9}+\left(\frac{2}{9}\right)^{2}+\dots\right)$$

$$+ \frac{5}{9}\left(1+\frac{5}{9}+\left(\frac{5}{9}\right)^{2}+\dots\right)$$

$$= \frac{4}{9}\cdot\frac{1}{1-\frac{2}{9}} + \frac{5}{9}\cdot\frac{1}{1-\frac{5}{9}}$$

$$= \frac{4}{9-2} + \frac{5}{9-5} = \frac{4}{7} + \frac{5}{4} = \underline{\underline{\frac{51}{28}}}$$

b) $\sum_{n=2}^{\infty} \frac{(-1)^n (2x)^n}{\sqrt{n}-1} = \sum_{n=2}^{\infty} (-1)^n a_n$

absolutna konvergenca: $\sum_{n=2}^{\infty} \frac{2^n |x|^n}{\sqrt{n}-1}$

kvocientni kriterij:

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{2^n \cdot 2 \cdot |x| \cdot |x|^n}{\sqrt{n+1}-1}}{\frac{2^n \cdot |x|^n}{\sqrt{n}-1}}$$

$$= \lim_{n\to\infty} \frac{2|x|(\sqrt{n}-1)}{\sqrt{n+1}-1} = 2|x| < 1$$

$\Rightarrow |x| < \frac{1}{2}$ ~~za~~ absolutna vrsta zagotovo konvergira

~~za~~ $|x| > \frac{1}{2}$ oz. $|2x| > 1$

je $\lim_{n\to\infty} (-1)^n a_n \neq 0$

$\Rightarrow$ vrsta divergira

pogledamo še krajišča intervala

$\left(-\frac{1}{2}, \frac{1}{2}\right)$

za $x = -\frac{1}{2}$ dobimo $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}-1}$

uporabimo Leibnizov kriterij:

- $\lim_{n\to\infty} \frac{1}{\sqrt{n}-1} = 0$ ✓

- <u>an padajoče</u>

$$\frac{1}{\sqrt{n+1}-1} \le \frac{1}{\sqrt{n}-1}$$

$$\sqrt{n}-1 \le \sqrt{n+1}-1$$

$$\sqrt{n} \le \sqrt{n+1}$$

$$n \le n+1 \ \checkmark$$

$\Rightarrow$ vrsta konvergira za $x = \frac{1}{2}$

za $x = -\frac{1}{2}$ dobimo vrsto

$$\sum_{n=2}^{\infty} \frac{(-1)^n(-1)^n}{\sqrt{n}-1} = \sum_{n=2}^{\infty} \frac{(-1)^{2n}}{\sqrt{n}-1}$$

$$= \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}.$$ Vrsta divergira, saj velja

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1} > \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} = \underbrace{\sum_{n=2}^{\infty} \frac{1}{n^{\frac{1}{2}}}}_{}$$

divergira

saj je vrsta

$\sum_{n=2}^{\infty} \frac{1}{n^{\alpha}}$ divergentna

za $\alpha \leq 1$

$\Rightarrow$ vrsta konvergira za

$x \in \left(-\frac{1}{2}, \frac{1}{2}\right]$