rešitve 2. kolokvija

MAT II

Naloga 1

$a > 2$

$$\arctan(ax) - \arctan(2x)$$

$$= \arctan(xy)\Big|_{y=2}^{y=a} = \int_2^a \frac{x}{1+(xy)^2}\,dy$$

$$\Rightarrow \int_0^\infty \frac{\arctan(ax) - \arctan(2x)}{x}\,dx$$

$$= \int_0^\infty dx \int_2^a \frac{\cancel{x}}{1+(xy)^2} \frac{1}{\cancel{x}}\,dy$$

$$f(x,y) = \frac{1}{1+(xy)^2}$$ zvezna na $[0,\infty) \times [a,2]$

Preverimo, da je integral

$$\int_0^\infty \frac{1}{1+(xy)^2}\,dx$$ enakomerno konvergenten

$$\left|\frac{1}{1+(xy)^2}\right| \le \frac{1}{1+(ax)^2}$$

$y \in [2, a]$

integral $\displaystyle\int_0^\infty \frac{1}{1+(ax)^2}\,dx$ pa konvergira

saj je

$t = ax$

$dt = a\,dx$

$$\frac{1}{a}\int_0^\infty \frac{dt}{1+t^2} = \frac{1}{a}\left[\arctan(t)\right]_0^\infty$$

$$= \frac{1}{a}\,\frac{\pi}{2} < \infty$$

$\Rightarrow$ lahko zamenjamo vrstni red integracije

$$\Rightarrow \int_2^a dy \int_0^\infty \frac{1}{1+(xy)^2}\,dx$$

$$= \int_2^a \frac{\arctan(xy)}{y}\Big|_0^\infty dy$$

$$= \int_2^a \frac{1}{y}\left(\frac{\pi}{2}\right) dy = \frac{\pi}{2} \ln y \Big|_2^a$$

$$= \frac{\pi}{2}(\ln a - \ln 2) = \underline{\underline{\frac{\pi}{2} \ln\left(\frac{a}{2}\right)}}$$

NALOGA 2

$$\int_1^\infty \frac{1}{x^2\sqrt{x-1}}\, dx = \int_1^0 \frac{-dt}{\sqrt{\frac{1}{t}-1}}$$

$$t = \frac{1}{x}$$

$$dt = -\frac{1}{x^2}\, dx$$

$$= \int_0^1 \frac{1}{\sqrt{\frac{1-t}{t}}}\, dt = \int_0^1 t^{\frac{1}{2}}(1-t)^{-\frac{1}{2}} dt$$

$$x - 1 = \frac{1}{2} \Rightarrow x = \frac{3}{2}$$

$$y - 1 = -\frac{1}{2} \Rightarrow y = \frac{1}{2}$$

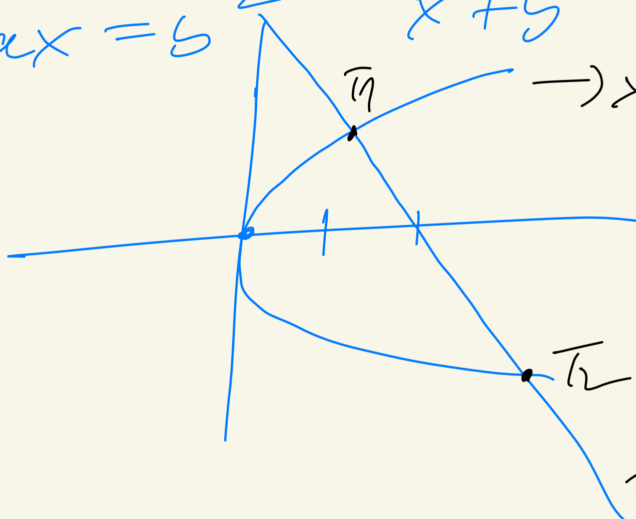
$$= B\left(\frac{3}{2}, \frac{1}{2}\right)$$

$$\Rightarrow B\left(\tfrac{3}{2}, \tfrac{1}{2}\right) = \frac{\Gamma\left(\tfrac{3}{2}\right)\Gamma\left(\tfrac{1}{2}\right)}{\Gamma(2)}$$

$$= \frac{\tfrac{1}{2} \cdot \sqrt{\pi} \cdot \sqrt{\pi}}{1} = \underline{\underline{\frac{\pi}{2}}}$$

NALOGA 3

$ax = y^2$ $\quad$ $x + y = 2a$



$\rightarrow x = \dfrac{y^2}{a}$

$\rightarrow x = 2a - y$

presečišče:

$$x = 2a - y \ \vee \ ax = y^2$$

$$a(2a - y) = y^2 \Rightarrow y^2 + ay - 2a^2 = 0$$

$$(y + 2a)(y - a) = 0$$

$\Rightarrow y_1 = -2a \Rightarrow x_1 = 2a - y_1$

$= 2a + 2a = 4a$

$\Rightarrow \boxed{T_2(4a, -2a)}$

și $y_2 = a \Rightarrow x_2 = 2a - y_2 = a$

$\Rightarrow \boxed{T_1(a, a)}$

uaugeu lk

$$\text{masa} = \mu = \rho \cdot \iint_D 1 \, dx\, dy$$

$$= \rho \int_{-2a}^{a} dy \int_{\frac{y^2}{a}}^{2a-y} 1 \, dx$$

$$= \rho \int_{-2a}^{a} \left(2a - y - \frac{y^2}{a}\right) dy$$

$$= \rho \left[2ay - \frac{y^2}{2} - \frac{y^3}{3a} \right]_{-2a}^{a}$$

$$= \rho \left(2a^2 - \frac{a^2}{2} - \frac{a^2}{3} - \left(-4a^2 - 2a^2 + \frac{8}{3}a^2 \right) \right)$$

$$= \rho a^2 \left(2 - \frac{1}{2} - \frac{1}{3} + 4 + 2 - \frac{8}{3} \right) = \underline{\underline{\frac{9}{2} \rho a^2}}$$

$$x_T = \frac{\rho \iint\limits_D x \, dx \, dy}{m}$$

$$\Rightarrow \iint\limits_D x \, dx \, dy = \int_{-2a}^{a} dy \int_{\frac{y^2}{a}}^{2a-y} x \, dx$$

$$= \int_{-2a}^{a} \left[\frac{x^2}{2} \right]_{\frac{y^2}{a}}^{2a-y} dy$$

$$=\frac{1}{2}\int_{-2a}^{a}\left((2a-y)^2-\frac{y^4}{a^2}\right)dy$$

$$=\frac{1}{2}\int_{-2a}^{a}\left(4a^2-4ay+y^2-\frac{y^4}{a^2}\right)dy$$

$$=\frac{1}{2}\left(4a^2y-2ay^2+\frac{y^3}{3}-\frac{y^5}{5a^2}\right)\Bigg|_{-2a}^{a}$$

$$=\frac{1}{2}\left(4a^3-2a^3+\frac{a^3}{3}-\frac{a^3}{5}\right.$$

$$\left.-\left(-8a^3-8a^3-\frac{8}{3}a^3+\frac{32}{5}a^3\right)\right)$$

$$=\frac{a^3}{2}\left(4-2+\frac{1}{3}-\frac{1}{5}+8+8+\frac{8}{3}-\frac{32}{5}\right)$$

$$=\frac{a^3}{2}\left(18+3-\frac{33}{5}\right)=\frac{36a^3}{5}$$

$$\Rightarrow x_T = \frac{\cancel{\rho} \cdot 36a^3 \cdot 2}{5 \cdot 9 \cdot \cancel{\rho} a^2} = \underline{a \cdot \frac{8}{5}}$$

$$y_T = \frac{\rho \cdot \iint_D y \, dx\, dy}{m}$$

$$\Rightarrow \iint_D y \, dx\, dy = \int_{-2a}^{a} y \, dy \int_{\frac{y^2}{a}}^{2a-y} dx$$

$$= \int_{-2a}^{a} y\left(2a - y - \frac{y^2}{a}\right) dy$$

$$= \int_{-2a}^{a} \left(2ay - y^2 - \frac{y^3}{a}\right) dy$$

$$= \left[ay^2 - \frac{y^3}{3} - \frac{y^4}{4a}\right]_{-2a}^{a}$$

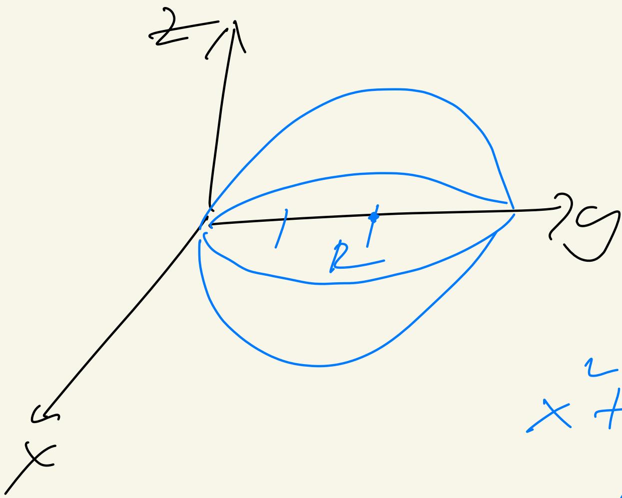
$$= a^3 - \frac{a^3}{3} - \frac{a^3}{4} - \left(4a^3 + \frac{8}{3}a^3 - 4a^3\right)$$

$$= a^3\left(1 - \frac{1}{3} - \frac{1}{4} - \frac{8}{3}\right) = -\frac{9}{4}a^3$$

$$\Rightarrow y_T = \frac{-9a^3\cancel{8}\cdot 2}{4\cdot 9\cancel{8}\cdot a^2} = -\frac{a}{2}$$

$\Rightarrow$ težišče ima koordinate $\left(\frac{8a}{5}, \frac{-a}{2}\right)$

NALOGA 4



sferične koordinate

$\varphi \in [0, \pi]$

$\vartheta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$r = ?$

$x^2 + y^2 - 2Ry + R^2 + z^2 \leq R^2$

$\Rightarrow r^2 \leq 2Ry$

$r^2 \leq 2R \cdot r \cdot \sin\varphi \cos\vartheta$

$r \leq 2R \cdot \sin\varphi \cos\vartheta$

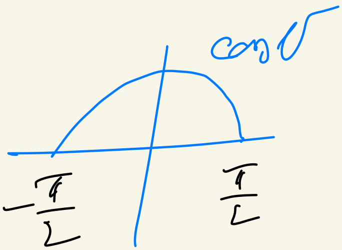
$$\Rightarrow \int_0^{\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\vartheta \int_0^{2R\sin\varphi\cos\vartheta} r^2 \cos\vartheta \cdot r\, dr$$

$$\int_0^{\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{r^4}{4}\cos\vartheta\right)\Bigg|_0^{2R\sin\varphi\cos\vartheta} d\vartheta$$

$$=4\int_0^{\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} R^4 \sin^4\varphi \cos^5\vartheta \, d\vartheta$$

integral: $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^5\vartheta \, d\vartheta$ $\quad 2x-1$

$$= 2\int_0^{\frac{\pi}{2}} \cos^5\vartheta \, d\vartheta$$

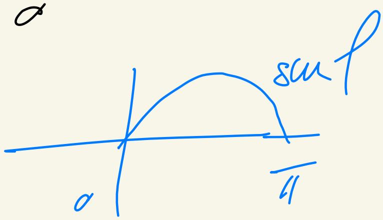


$$= B\left(3, \frac{1}{2}\right) = \frac{\Gamma(3)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{2}\right)}$$

$$= \frac{2\cdot\sqrt{\pi}}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}} = \underline{\underline{\frac{16}{15}}}$$

integral:

$$\int_0^{\pi} \sin^4\varphi \, d\varphi = 2\int_0^{\frac{\pi}{2}} \sin^4\varphi \, d\varphi$$



$$= B\left(\frac{1}{2}, \frac{5}{2}\right)$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(3)} = \frac{\sqrt{\pi}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{2}$$

$$= \frac{3\pi}{8}$$

$\Rightarrow$ daraus $4R^4 \cdot \frac{3\pi}{8} \cdot \frac{16}{15} = \boxed{\frac{8R^4\pi}{5}}$