



DISCRETE MATHEMATICS 2

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I. PERMUTATION GROUPS

Exercises.

- I.1 Show that if X and Y are (not necessarily finite) sets with $|X| = |Y|$, then $S_X \cong S_Y$.
- I.2 Let X be a set.
 - (a) If $|X| = n$, how many elements does the set S_X have?
 - (b) Let X be a countably infinite set, that is, $|X| = |\mathbb{N}|$. Prove that $|S_X| \geq |\mathbb{N}|$ (that is, strictly greater than $|\mathbb{N}|$). Can you determine $|S_X|$?
- I.3 Prove that a k -cycle $\sigma = (x_1 \cdots x_k)$ and an ℓ -cycle $\tau = (y_1 \cdots y_\ell)$, both elements of S_n , are equal if and only if $k = \ell$ and for some $b \in \mathbb{Z}$, $x_{i+b} = y_i$ or every $1 \leq i \leq r$ (the indices are taken modulo r).
- I.4 Prove that every permutation σ can be written as a product of disjoint cycles. **Hint:** two symbols $x, y \in X$ lie in the same cycle of σ if some power of σ maps x to y . Prove that this condition defines an equivalence relation and hence a partition of X .
- I.5 Prove that disjoint cycles commute.
- I.6 Let $n \in \mathbb{N}$.
 - (a) Prove that if $\sigma = (x_1 \cdots x_k)$ is a k -cycle and $\mu \in S_n$ then $\mu\sigma\mu^{-1} = (\mu(x_1) \cdots \mu(x_k))$.
 - (b) Show that for every $k \leq n$ every two k -cycles are conjugate.
 - (c) Conclude that two permutations in S_n are conjugate if and only if they have the same cycle-type.

- I.7 Let X be an infinite set. An *infinite cycle* in S_X is a permutation γ such that there exists a family $Z = \{x_i : i \in \mathbb{Z}\}$ of elements of X with $\gamma(x_i) = x_{i+1}$ for every $i \in \mathbb{Z}$ and $\gamma(y) = y$ for every $y \in X \setminus Z$. Find two infinite cycles in $S_{\mathbb{Z}}$ that are not conjugate.
- I.8 Let $n \in \mathbb{N}$. Show that the set of involutions $I = \{(1\ k) : 2 \leq k \leq n\}$ is a minimal generating set of S_n . That is, show that

$$S_n = \langle (1\ k) : 2 \leq k \leq n \rangle,$$

and that no proper subset of I generates S_n .

- I.9 Let $n \in \mathbb{N}$. Show that The involution $(1\ n)$ and the n -cycle $(1\ 2\ \cdots\ n)$ generate S_n .
- I.10 Let $n \in \mathbb{N}$, find necessary and sufficient conditions for $i, j \in [n]$ so that

$$\langle (i\ j), (1\ 2\ \cdots\ n) \rangle = S_n$$

- I.11 Show that the group $S_{\mathbb{N}}$ cannot be generated by a finite number of permutations.
- I.12 Prove that the set A_n is a subgroup of S_n .
- I.13 Prove that an r -cycle in S_n is even if and only if r is odd. Conclude that a permutation σ is even if and only if the number of even entries in its cycle-type is even.
- I.14 Let $n \geq 3$ and let A_n denote the alternating group.

- Show that

$$A_n = \langle (x\ y\ z) : x, y, z \in [n] \rangle.$$

- Show that

$$A_n = \langle (1\ 2\ z) : z \in [n] \rangle.$$

- Show that if x and y are fixed elements in $[n]$ then

$$A_n = \langle (x\ y\ z) : z \in [n] \rangle.$$

- I.15 Prove that if Γ is a permutation group, then either Γ consists of only even permutations or half of the permutations in Γ are even. Conclude that A_n is normal in S_n and that every permutation group that contains an odd permutation has a normal subgroup of index 2.
- I.16 Show that if n is prime, then any two cyclic of order n in S_n are conjugate. Find two cyclic groups of order 6 in S_6 that are not conjugate.
- I.17 Let $n \in \mathbb{N}$ and $D_n = \langle \rho, \sigma \rangle$ the dihedral group defined in ??.
- (a) Show that these permutations satisfy the following relations:

$$\rho^2 = \varepsilon$$

$$\sigma^n = \varepsilon$$

$$\rho\sigma\rho = \sigma^{-1}$$

(b) Define $\tau = \sigma\rho$. Show that the relations above are equivalent to

$$\tau^2 = \rho^2 = (\tau\rho)^n = \varepsilon$$

We will prove later that any group generated by two involutions is isomorphic to a dihedral group.

- 1.18 Prove that $|D_n| = 2n$. **Hint:** work the cases where n is even and n is odd separately.
- 1.19 Find the conjugacy classes of the symmetric group S_5 and of the alternating group A_5 . Hence, show that A_5 is the only normal subgroup of S_5 (apart from 1 and S_5 , and that A_5 is simple.)
- 1.20 If $H \leq G$ are groups, the *normaliser of H in G* is the largest subgroup of G in which H is normal. Find the normaliser in S_n of the cyclic group $C_n = \langle (1\ 2\ \dots\ n) \rangle$.