1 Background and Axioms

Georg Cantor famously proved that the set $\mathbb{N}$ of natural numbers and the set $\mathbb{R}$ of real numbers cannot be put in bijection with each other. Whereas the set $\mathbb{N}$ is **countable**, the set $\mathbb{R}$ is **uncountable**. The infinity of real numbers is strictly larger than the infinity of natural numbers. In contemplating these different sizes of infinity, Cantor further suggested that there should be no intermediate size of infinity between $\mathbb{N}$ and $\mathbb{R}$, formulating this as follows.

The continuum hypothesis (CH)

Every infinite set of real numbers can be put in bijection with either $\mathbb{N}$ *or* $\mathbb{R}$*.*

Cantor was unable to he prove this statement, and so left it as a *hypothesis*. The brilliant and influential mathematician David Hilbert, who thought it was an important question, posed the continuum hypothesis as the first problem on his famous list of mathematical challenges for the 20th century, presented at the 1900 International Congress of Mathematicians. Even today, the continuum hypothesis continues to be a question that divides opinion as to whether it is in reality true, false or perhaps even possesses no objective truth value, allowing its truth or falsity to be chosen according to preference.

A second and related contribution of Cantor's was the identification of the general notion of **cardinal**, a quantity measuring the size of sets, and the $\aleph$ (aleph) notation that is used as its numbering system. The natural numbers $\mathbb{N}$ have the smallest infinite cardinality $\aleph_0$, which is the only countable infinite cardinality. The next aleph cardinal $\aleph_1$ can be defined as the cardinality of the smallest uncountable **well-order**. And the uncountable cardinality of the real numbers $\mathbb{R}$ can be expressed as $2^{\aleph_0}$, using the operation of **cardinal exponentiation**. Exploiting this notation, we can formulate a version of the continuum hypothesis as an equality in **cardinal arithmetic**.

The aleph continuum hypothesis (ACH)

$$2^{\aleph_0} \;=\; \aleph_1$$

We call the above statement the **aleph continuum hypothesis** because it is formulated using the aleph notation. The relationship between Cantor's original CH and its aleph version is slightly subtle. Although one implication, that from ACH to CH, is straightforward, the converse implication requires the **axiom of choice**. For this reason, we find it useful to give ACH its own name.[1] Without the axiom of choice, it is not even possible to show that $\aleph_1 \leq 2^{\aleph_0}$.

Cardinal arithmetic is in essence the arithmetic of infinite sizes. It is not, however, the only arithmetic of infinite quantities. Another is **ordinal arithmetic**: the arithmetic of well-orderings.

Both cardinal and ordinal arithmetic are very much the province of **set theory**. Set theory is a fascinating field of mathematics, which impinges on several mainstream mathematical

[1] In the literature, often the axiom of choice is assumed, and both CH and ACH are referred to interchangeably as CH.

areas, and also has deep connections with the philosophy of mathematics. Cardinal and ordinal arithmetic form just a part of set theory. This course will study set theory more broadly, including cardinal and ordinal arithmetic amongst other equally interesting topics. Nevertheless, cardinal arithmetic will provide us with a focal point to which we shall return at several points during the course, each time with new tools to hand.

There are (at least) four different sides to set theory in relation to the rest of of mathematics.

1. Set theory as a complement to other mathematical subjects.

   Sets were originally introduced as a tool for organising and defining concepts in other mathematical fields. Indeed, crucial use of the notion of set is made in many mathematical areas. In several such fields there is also a dependency of central mathematical results on basic assumptions (axioms) about sets; for example, many theorems of mathematics rely on the axiom of choice.

2. Set theory as a foundation for mathematics.

   Once the notion of set has been introduced together with appropriate axioms governing its use a truly remarkable situation arises. It turns out that all the common notions of ordinary mathematics (the different kinds of number, functions, etc.) can be redefined purely in terms of sets. Furthermore, all the expected properties of these notions, and indeed all the theorems of mathematics, can be proved to follow from the few axioms for set theory alone. Thus set theory provides one possible *foundation* for the entirety of mathematics. All of mathematics can ultimately be reduced to definitions and proofs in set theory. It is a truly astonishing fact that an area as immense as mathematics can be reduced to such a minimal conceptual basis.

3. Set theory as a source of new mathematical areas.

   Not only does set theory subsume the standard mathematical fields, but the abstractions of set theory naturally suggest new subjects for mathematical investigation. In particular, the mathematics of infinite quantities emerges naturally from set theory, leading to mathematical theories of cardinals and ordinals. More generally, set theory naturally accommodates infinite versions of other mathematical structures such as trees and graphs, leading to a rich mathematical theory of infinite combinatorics.

4. Set theory as a basis for independence proofs.

   The reduction of mathematics to set theory mentioned in point 2 above opens up a new possibility: techniques from mathematical logic can be used to prove that mathematical statements are **independent** from the axioms for set theory — that is, neither the statement nor its negation can be proved from the axioms. The most famous example of this phenomenon is the continuum hypothesis CH itself.[2] In 1940, Kurt Gödel showed that $\neg$CH is not provable from the axioms of set theory. In 1963, Paul Cohen showed also that CH is also not provable.[3] Arguably, these results settle the status of Hilbert's first problem, mentioned earlier, in that they show that there is no hope of either proving or refuting CH using the usual axioms for mathematics, namely the axioms

[2] Similarly the aleph continuum hypothesis (ACH).

[3] For this achievement, Cohen was awarded a Fields Medal in 1966 (alongside Michael Atiyah, Alexander Grothendieck and Stephen Smale).

for set theory. There is disagreement, however, about whether this really solves the continuum hypothesis as a mathematical question. Some believe that CH is either true of false in reality, and we have simply not yet found the right axioms to settle it one way or the other.[4]

In this course, we shall start of with aspect 1 above, and consider sets as they arise as an abstraction needed to support other pre-existing mathematical fields. This leads to a natural set of axioms. It will turn out that these axioms are sufficiently rich to derive existing mathematics as a consequence of the axioms, thus addressing aspect 2. Furthermore, the axioms give rise to rich theories of cardinals and ordinals, thus providing a glimpse of aspect 3. The theory of cardinals will also provide us with a plentiful supply of properties that will turn out to be independent of the axioms, providing us with several examples illustrating aspect 4. We will not, however, have time to introduce the proof methods that are used to justify such independence claims, for these are highly technical and require a firm grounding in mathematical logic that will not be assumed in this course.

1.1 Sets in mathematics

As we learn mathematics, the kinds of mathematical entities we work with become progressively more complicated.

- Numbers: integers, rationals, reals, complex, ...
- Constructions: tuples $(x_1, \ldots, x_n)$, functions, indexed families $\{x_i\}_{i \in I}$, ...
- Sets: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{N} \times \mathbb{R}$, $\mathbb{R}^n$, $C^n(\mathbb{R}^k, \mathbb{R})$, ...
- Structures: ordered sets, graphs, algebraic structures (e.g., groups, rings, fields), ...
- Sets of sets: open sets $\mathcal{O}(\mathbb{R})$, Borel sels $\mathcal{B}(\mathbb{R})$, powersets $\mathcal{P}(\mathbb{N})$, ...
- Sets of sets of sets: the set of all topologies on $\mathbb{R}$, ...
- etc.

Mathematical experience shows that sets are an important notion, and that it is important to be able to allow sets to contain other sets as their elements.

In the remainder of today's lecture, we aim to provide axioms for sets as they are actually used in mathematics. In mathematics, one starts with simple mathematical entities such as number and function, and the sets are later introduced alongside such preexisting entities. It is thus natural to proceed in the same way. We shall work with a 'universe' of mathematical entities that is allowed to include numbers and other mathematical constructs as primitive entities that need not themselves be sets. Our aim is to understand the role sets play in such a universe.

It should be said that the approach outlined above differs from that offered in most textbooks on set theory in one respect. Typically, textbooks make the assumption that sets

[4] For an interesting and forthright argument in this direction, by one of the great mathematicians of the 20th century, I recommend reading the article *What is Cantor's Continuum Problem* by Kurt Gödel. Although this article was written before Cohen's proof of the independence of CH, it was written with the expectation that CH would be found to be independent.

are the only mathematical entities that exist. We believe our approach offers a more natural starting point, as it better concords with mathematical practice. Quite swiftly, however, we shall find that standard mathematical constructs such as number and function can be redefined as sets. From then on, it will be immaterial whether or not the universe contains entities in it that are not themselves sets.

1.2 Sets and classes

We introduce the notation $\underline{U}$ for the (perhaps open-ended) **universe** of all mathematical entities that we are allowed to use as elements of sets. We note the following.

- Sets are collections of elements from $\underline{U}$.
- Since sets can be elements of sets, they are themselves elements of $\underline{U}$.

Indeed, we take the two points above as the definition of the concept of set.

A **set** is a collection of elements from $\underline{U}$ that is itself an element of $\underline{U}$.

The setting is thus far *very* general. We have not yet imposed any axioms on sets. Nevertheless, even at this generality it is possible to use Betrand Russell's famous argument, **Russell's paradox**, to show that we must not allow every collection of elements from $\underline{U}$ to be considered a set. In order to do this, we first introduce some notation and terminology.

We call an arbitrary collection of elements from $\underline{U}$ a **class**. Typically, we shall use underlined capital letters $\underline{X}, \underline{Y}, \underline{Z}, \underline{A}, \underline{B}, \underline{C}, \ldots$ to denote classes. Also, For any property $P(x)$ of elements x of $\underline{U}$ we introduce the notation:

$$\{x \mid P(x)\} \;:=\; \text{the collection of all } x \in \underline{U} \text{ that satisfy } P\,.$$

For a class $\underline{X}$ and an arbitrary element $x \in \underline{U}$ we define:

$$x \in \underline{X} \;\equiv\; \text{the element } x \text{ belongs to } \underline{X}\,.$$

It follows in particular that, for any $y \in \underline{U}$,

$$y \in \{x \mid P(x)\} \;\Leftrightarrow\; P(y)\,. \tag{1}$$

Theorem 1.1 (Russell's paradox)**.** *Not every class is a set.*

Proof. Define the class

$$\underline{R} \;:=\; \{x \mid x \text{ is a set and } x \notin x\}\,.$$

Suppose for contradiction that $\underline{R}$ is a set. Then $\underline{R} \in \underline{U}$ and, by (1), we have that

$$\underline{R} \in \underline{R} \;\Leftrightarrow\; \underline{R} \notin \underline{R}\,.$$

This is a contradiction. Hence $\underline{R}$ is not a set. □

A class that is not a set is called a **proper class**. So the class $\underline{R}$ defined in the proof above is an example of a proper class.

As we have seen above, unavoidably there exist classes that are not sets. It is thus incumbent on us to provide some principles governing which classes we allow to form sets. We shall formulate such principles as a collection of **axioms for sets**. Before coming to these, it is useful to first explore important various important mathematical concepts pertaining to classes that do not depend on the notion of set.

1.3 Basic notions related to classes

It is implicit in the very notion of collection itself that collections are determined by their elements. We formulate this an

The extensionality axiom (for classes)

If two classes $\underline{X}$ and $\underline{Y}$ have the same elements then they are equal. Symbolically:

$$\forall x\ (x \in \underline{X} \Leftrightarrow x \in \underline{Y}) \ \Rightarrow\ \underline{X} = \underline{Y}\,.$$

In the above axiom and henceforth, unbounded quantifiers like $\forall x$ always range over the universe $\underline{U}$; that is $\forall x$ means $\forall x \in \underline{U}$.

For classes $\underline{X}, \underline{Y}$, we introduce the notation

$$\underline{X} \subseteq \underline{Y} \ \equiv\ \forall x \in \underline{U}\ (x \in \underline{X} \Rightarrow x \in \underline{Y})\,,$$

in which case we say that $\underline{X}$ is a **subclass** of $\underline{Y}$. We can use a property $P(x)$ of elements of $\underline{U}$ to define a subclass of any class $\underline{X}$:

$$\{x \in \underline{X} \mid P(x)\} \ :=\ \{x \mid x \in \underline{X}\ \wedge\ P(x)\}\,.$$

In mathematics, it is frequently useful to consider a basic notion of transformation (or morphism) between mathematical structures. The appropriate notion of transformation between classes is that of a **class function**.

A property $F(x, y)$ of elements x, y of $\underline{U}$ is said to define a **class function** from a class $\underline{X}$ to a class $\underline{Y}$ if:

1. $\forall x \in \underline{X}\ \exists! y \in \underline{Y}\ F(x, y)$, and
2. $\forall x, y\ F(x, y) \Rightarrow x \in \underline{X} \wedge y \in \underline{Y}$.

Here $\exists!$ is the *unique-existence* quantifier. Thus property 1 above can be equivalently formulated as:

$$\forall x \in \underline{X}\ \exists y \in \underline{Y}\ F(x, y) \quad \text{and} \quad \forall x \in \underline{X}\ \forall y, z \in \underline{Y}\ \ F(x, y) \wedge F(x, z)\ \Rightarrow\ y = z\ \ .$$

We write $F \colon \underline{X} \to \underline{Y}$ to declare that F is a class function from $\underline{X}$ to $\underline{Y}$. We shall use standard notation and terminology for such functions. Given $x \in \underline{X}$, we write $F(x)$ for the unique y

such that $F(x,y)$. Given class functions $F\colon \underline{X} \to \underline{Y}$ and $G\colon \underline{Y} \to \underline{Z}$ there is a composite class function $G \circ F\colon \underline{X} \to \underline{Z}$ satisfying $(G \circ F)(x) = G(F(x))$.[5]

We say that F is **injective** (or **one-to-one**) if, for all $x, x' \in \underline{X}$, it holds that $F(x) = F(x')$ implies $x = x'$. We say that F is **surjective** (or **onto**) if, for every $y \in \underline{Y}$, there exists $x \in \underline{X}$ such that $F(x) = y$. We say that F is **bijective** (or **one-to-one onto**, or an **isomorphism**) if it is both injective and surjective. F is bijective if and only if there exists a (necessarily unique) class function $F^{-1}\colon \underline{Y} \to \underline{X}$ such that $F^{-1} \circ F$ is the identity on $\underline{X}$ and $F \circ F^{-1}$ is the identity on $\underline{Y}$.

The **image** class of a class function F is defined by:

$$\mathsf{image}(F) \;:=\; \{y \in \underline{Y} \mid \exists x \in \underline{X}\ (F(x) = y)\}\,.$$

1.4 Axioms for sets

In order to appreciate the axioms, it is worth having some intuition about what it should mean for a class to be a set. There are different possible intuitions about this, all of them reasonable, and all mutually compatible.

- A set is a class that is, in some sense, *small* in comparison with the universe $\underline{U}$.
- A set is a class that is definite in scope; i.e., it is not open-ended in the way that the universe may be considered to be.
- A set must be built up in a principled way out of elements of the universe that have been previously defined independently of the set under construction.

Whatever intuition one prefers, the following axioms for sets should seem reasonable. All of them have a common flavour: they assert that certain specified classes are sets.

The separation axiom

If X is a set then, for any property $P(x)$ of elements of $\underline{U}$, the class $\{x \in X \mid P(x)\}$ is a set.

The pairing axiom

For any $x, y \in \underline{U}$, the class

$$\{x, y\} \;:=\; \{z \mid z = x \ \vee\ z = y\}$$

is a set.

We call the set $\{x, y\}$ the **unordered pair** of x and y. As a special case, for any $x \in \underline{U}$, we have a **singleton set** $\{x\} := \{x, x\}$.

If $\underline{X}$ is a **class of sets**, that is a class all of whose elements are sets, then we define the notation:

[5] Classes and class functions form a *category*, except that one needs to be careful about what this means. Classes are not in general elements of $\underline{U}$ (Russell's paradox), so cannot in general themselves be used as elements of classes. There is therefore no class of all classes. So the collection of objects of the category of classes does not itself form a class. One way of thinking about such a 'superlarge' category is that objects and morphisms of the category can be understood individually, but one should not try to contemplate the collection of all of them at once.

$$\bigcup \underline{X} \ := \ \{x \mid \exists X \in \underline{X} \ (x \in X)\} \,.$$

The union axiom

For any set X of sets, the class $\bigcup X$ is a set.

If X is a set and $X \subseteq \underline{Y}$, we say that X is a **subset** of the class $\underline{Y}$. For any class $\underline{Y}$, consider the derived class:

$$\mathcal{P}(\underline{Y}) \ := \ \{X \mid X \text{ is a set } \wedge \ X \subseteq \underline{Y}\} \,.$$

That is $\mathcal{P}(\underline{Y})$ is the *class of all subsets of* $\underline{Y}$. We call $\mathcal{P}(\underline{Y})$ the **powerclass** of $\underline{Y}$.

The powerset axiom

For any set X, the class $\mathcal{P}(X)$ is a set.

This axiom justifies the terminology of calling $\mathcal{P}(X)$, for a set X, the **powerset** of X.

The next axiom makes use of the notion of class function.

The replacement axiom

For any class function $F \colon X \to \underline{U}$, whose domain X is a set, the image class $\mathsf{image}(\underline{F})$ *is itself a set.*

1.5 Cartesian product

The main property of set theory that justifies its role as a foundation for mathematics is that the entirety of the body of mathematics can be derived from the set-theoretic axioms. We begin by showing one very simple case of this: the construction of cartesian products.

Standard set-theoretic texts typically proceed as follows. A concrete construction of product sets is given using **Kuratowski pairs**. This construction is then used as the definition of product, and ultimately all mathematical results about products are really, if all definitions are fully expanded, results about sets of Kuratowski pairs. However, the fact that the product was defined in this particular way (rather than using a different concrete construction) of course never plays any further role (at least not in any sensibly developed body of mathematics).

We prefer to proceed in a different, and perhaps more modern way. We define the notion of product by *characterising* it independently of any particular concrete construction. All subsequent mathematical development that uses products will then make use of the abstract characterisation only. This approach ensures that the mathematics developed from set theory follows a style that is 'kosher'. It is impossible for the mathematics we develop to depend on accidental features of a particular chosen concrete definition of product. Under our approach, a concrete construction is used for one thing only. It is used to prove that the existence of products can be proved as a consequence of the set-theoretic axioms. Having thus done its job, the concrete construction is never made use of again.

Another way in which we differ from texts on set theory is that we define, in the first place, the notion of product for classes, from which we derive product sets as a special case.

A **product** of two classes $\underline{X}$ and $\underline{Y}$ is a class $\underline{X} \times \underline{Y}$ equipped with class functions $\pi_1 \colon \underline{X} \times \underline{Y} \to \underline{X}$ and $\pi_2 \colon \underline{X} \times \underline{Y} \to \underline{Y}$ (the **projections**) that together satisfy:

$$\forall x \in \underline{X}\ \forall y \in \underline{Y}\ \exists! z \in \underline{X} \times \underline{Y}\ \ \pi_1(z) = x \wedge \pi_2(z) = y\ \ .$$

We write (x, y) for the unique z such that $\pi_1(z) = x$ and $\pi_2(z) = y$.

Proposition 1.2. *If $(\underline{X} \times \underline{Y}, \pi_1, \pi_2)$ is a product then, for any class $\underline{Z}$ and class functions $F \colon \underline{Z} \to \underline{X}$ and $G : \underline{Z} \to \underline{Y}$, there exists a unique class function $H \colon \underline{Z} \to \underline{X} \times \underline{Y}$ such that $\pi_1 \circ H = F$ and $\pi_2 \circ H = G$.*

Proposition 1.3. *If $(\underline{X} \times \underline{Y}, \pi_1, \pi_2)$ and $(\underline{X} \times' \underline{Y}, \pi'_1, \pi'_2)$ are two products then there exists an isomorphism $I \colon \underline{X} \times \underline{Y} \to \underline{X} \times' \underline{Y}$ such that $\pi'_1 \circ I = \pi_1$ and $\pi'_2 \circ I = \pi_2$. (Hence also $\pi_1 \circ I^{-1} = \pi'_1$ and $\pi_2 \circ I^{-1} = \pi'_2$.)*

Proposition 1.4. *If $X \times Y$ is a product of two sets X and Y then $X \times Y$ is itself a set.*

Theorem 1.5. *A product $(\underline{X} \times \underline{Y}, \pi_1, \pi_2)$ exists, for any two classes $\underline{X}$ and $\underline{Y}$.*

It is in the proof of the above theorem that we make use of Kuratowski pairs, which are defined by a repeated application of unordered pairs:

$$(x, y)_K\ :=\ \{\{x\}, \{x, y\}\}\ ,$$

which is well-defined as a class, by the pairing axiom, and is furthermore itself a set, by another application of the same axiom. Thus $(x, y)_K$ is an element of $\underline{U}$. The crucial property of this construction that characterises ordered pairs is:

Proposition 1.6. *For all $x, y, z, w \in \underline{U}$,*

$$(x, y)_K = (z, w)_K\ \Rightarrow\ x = z\ \wedge\ y = w\,. \tag{2}$$

For classes $\underline{X}, \underline{Y}$, we define the **Kuratowski product class**

$$\underline{X} \times_K \underline{Y}\ :=\ \{(x, y)_K \mid x \in \underline{X}, y \in \underline{Y}\}\,.$$

Here, the right-hand expression is a convenient abbreviation for

$$\{z \mid \exists x \in \underline{X}, y \in \underline{Y}\ z = (x, y)_K\}\,.$$

By (2), for every $z \in \underline{X} \times_K \underline{Y}$, there exist unique $x \in \underline{X}$ and $y \in \underline{Y}$ such that $z = (x, y)_K$. The mappings $z \mapsto x$ and $z \mapsto y$ the respectively define the projections π_1 and π_2 required to prove Theorem 1.5.

Having proved the theorem, we henceforth put the Kuratowski product definition to one side. Although we shall make copious use of products in this course, all we assume is that we have an arbitrary product structure according to the boxed definition at the start of the section. We shall never require that the product is given by any one specific construction.

Using product classes, we can define the **graph** of a class function $F \colon \underline{X} \to \underline{Y}$ as the subclass of $\underline{X} \times \underline{Y}$ given by:

$$\underline{\mathsf{gr}}(F)\ :=\ \{(x, y) \in \underline{X} \times \underline{Y} \mid F(x) = y\}\ \ .$$

Proposition 1.7. *If $F\colon X \to \underline{Y}$ is a class function whose domain X is a set then the graph $\underline{\mathsf{gr}}(F)$ is also a set.*

Henceforth, we shall call class functions whose domains are sets simply **functions**, and we shall typically use lower case letters for them. Since the graph of any such $f\colon X \to \underline{Y}$ is a set, we also omit the underlining on the graph notation, writing $\mathsf{gr}(f)$.

1.6 Function spaces

For any set X and class $\underline{Y}$, the collection of all functions from X to $\underline{Y}$ itself forms a class $\underline{Y}^X$, called an **exponential** (or **function space**). Once again, we adapt a modern style of presentation, and define the class $\underline{Y}^X$ in terms of its characterising properties, rather than by committing ourselves to using any one particular representation of functions for the elements of $\underline{Y}^X$.

Given a set X and class $\underline{Y}$ an **exponential** $\underline{Y}^X$ (or $X \to \underline{Y}$) is a class equipped with a class function $\epsilon\colon \underline{Y}^X \times X \to \underline{Y}$ (the **evaluation** function) satisfying: for any function $f\colon X \to \underline{Y}$ there exists a unique $g \in \underline{Y}^X$ such that

$$\forall x \in X \quad \epsilon(g, x) = f(x) \quad .$$

We write $\ulcorner f \urcorner$ for the unique g satisfying the property above.

Proposition 1.8 (Currying)**.** *If $(\underline{Y}^X, \epsilon)$ is an exponential then, for any class $\underline{Z}$ and class function $f\colon \underline{Z} \times X \to \underline{Y}$ there exists a unique class function $g\colon \underline{Z} \to \underline{Y}^X$ such that*

$$\forall z \in \underline{Z}\ \forall x \in X \quad \epsilon(g(z), x) = f(z, x) \quad .$$

Proposition 1.9 (Currying)**.** *If $(X \to \underline{Y}, \epsilon)$ and $(X \to' \underline{Y}, \epsilon')$ are both exponentials, then there exists an isomorphism $I\colon (X \to \underline{Y}) \to (X \to' \underline{Y})$ such that $\epsilon' \circ I = \epsilon$. (Hence also $\epsilon \circ I^{-1} = \epsilon'$.)*

Proposition 1.10. *If Y^X is an exponential of two sets X and Y then Y^X is itself a set.*

Theorem 1.11. *An exponential $(\underline{Y}^X, \epsilon)$ exists, for any set X and class $\underline{Y}$.*

Again, the theorem is proved by giving a concrete definition of the exponential $\underline{Y}^X$, which we define as the class of graphs of functions.

$$X \to_G \underline{Y} \ := \ \{g \in \mathcal{P}(X \times \underline{Y}) \mid \forall x \in X\ \exists! y \in \underline{Y}\ (x, y) \in g\} \ ,$$

The corresponding evaluation function $\epsilon_G\colon (X \to_G \underline{Y}) \times X \to \underline{Y}$ is given by

$$\epsilon_G(g, x) \ \mapsto \ \text{the unique } y \text{ such that } (x, y) \in g \ .$$

It is routine to verify that this data indeed defines an exponential.