Kardinalna aritmetica 2

A natural number structure is a triple $(\underline{N}, s, 0)$ where:

$\underline{N}$ is a class

$0 \in \underline{N}$

$s : \underline{N} \to \underline{N}$ is a class function

Satisfying:

injectivity: $\forall x, y \in \underline{N} \quad s(x) = s(y) \Rightarrow x = y$

acyclicity: $\forall x \in \underline{N} \quad s(x) \neq 0$

induction: if a class $\underline{X}$ satisfies:

$$0 \in \underline{X}$$

and $\forall x \in \underline{N} \quad x \in \underline{X} \Rightarrow s(x) \in \underline{X}$

then $\underline{N} \subseteq \underline{X}$

The recursion theorem

Suppose $(\underline{N}, s, 0)$ is a natural number structure.

Given any triple $(\underline{A}, F, a)$, where $\underline{A}$ is a class

$F: \underline{A} \to \underline{A}$ is a class function and $a \in \underline{A}$,

there exists a unique class function $R: \underline{N} \to \underline{A}$ satisfying

$$R(0) = a \qquad \forall n \in \underline{N} \quad R(s(n)) = F(R(n))$$

(We call a function R satisfying the two properties above a homomorphism).

<u>Corollary</u> If $(\underline{N}, s, 0)$ and $(\underline{N}', s', 0')$ are two natural number structures then they are isomorphic. Indeed the unique homomorphisms $I: \underline{N} \to \underline{N}'$ and $I': \underline{N}' \to \underline{N}$ are such that $I' \circ I$ and $I \circ I'$ are the identities on $\underline{N}$ and $\underline{N}'$ respectively.

The parameterised recursion theorem

Suppose $(\underline{N}, s, 0)$ is a natural number structure.

Given any class $\underline{Z}$ and any triple $(\underline{A}, F, B)$, where $\underline{A}$ is a class

$F : \underline{Z} \times \underline{A} \to \underline{A}$ and $B : \underline{Z} \to \underline{A}$ are class functions,

there exists a unique class function $R : \underline{Z} \times \underline{N} \to \underline{A}$ satisfying

$\forall z \in \underline{Z} \quad R(z, 0) = B(z) \qquad \forall n \in \underline{N} \quad R(z, s(n)) = F(z, R(z,n))$

Example (addition)

Consider $\underline{A} := \underline{N} \qquad Z := \underline{N} \qquad$ with

$$B(z) = z \qquad B : \underline{N} \to \underline{N}$$

$$F(z,n) = s(n) \qquad F : \underline{N} \times \underline{N} \to \underline{N}$$

Then $\underline{R} : \underline{N} \times \underline{N} \to \underline{N}$ satisfies

$$R(z, 0) = z$$

$$R(z, s(n)) = s(R(z,n))$$

These properties characterise addition

So we write $x + y$ for $R(x, y)$

Theorem If the universe is nonempty then a natural number structure exists.

Idea: We define a concrete natural number structure by choosing particular sets as representatives for individual natural numbers (the <u>von Neuman numbers</u>)

The class of all such von Neuman numbers can then be shown to be a natural number structure.

A set X of sets is a <u>von Neumann number</u> (vN number) if:

(1) for every subset $Y \subseteq X$ satisfying

$$\emptyset \in X \Rightarrow \emptyset \in Y$$

and $\forall z \in Y \; (z^+ \in X \Rightarrow z^+ \in Y)$

it holds that $Y = X$

(2) $X = \emptyset$ or $\exists z \in X \; (X = z^+)$

For a set z we define $z^+ = z \cup \{z\}$

Define

$$\underline{\mathrm{vN}} := \{ X \mid X \text{ is a set of sets and } X \text{ is a vN number} \}$$

<u>Theorem</u> $(\underline{\mathrm{vN}}, (-)^+, \emptyset)$ is a natural number structure.

The axiom of infinity

We say that a set X is <u>infinite</u> if there exists an injective function from X to itself that is not surjective.

Proposition The following are equivalent

1. There exists an infinite set
2. There exists a set I of sets such that $\emptyset \in I$ and $\forall x \in I\,(x^+ \in I)$
3. The class $\underline{vN}$ is a set

Furthermore if $(\underline{N}, s, 0)$ is any natural number structure then the following is also equivalent

4. The class $\underline{N}$ is a set.